THE GRADUATION OF OBSERVATIONAL DATA

by

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INTRODUCTORY.

The present thesis takes as its starting-point the two papers in which Professor E.T.Whittaker has recently published a new theory* of Graduation. The basic principle introduced by Professor Whittaker is that the problem of Graduation belongs essentially to the mathematical theory of Probability; as he says, ' we have the given observations, and they would constitute the "most probable" values of u for the corresponding values of the argument, were it not that we have à priori grounds for believing that the true values of u form a smooth series, the irregularities being due to accidental causes which it is desirable to eliminate. The problem is to combine all the materials of judgment— the observed values and the à priori considerations — in order to obtain the resulting "most probable" values of u'. By the methods of Inductive Probability Professor Whittaker arrives at formulae of graduation which, from the theoretical point of view, appear to be of the highest interest and value: unfortunately they are (for reasons developed below) not well adapted for practical computation and in fact are altogether impossible if the number of data to be graduated is large: and the aim of the present thesis is to overcome this defect by transforming Professor Whittaker's theory into a form in which all the operations required can be readily carried out, and the

[* "On a new method of Graduation", Proc.Edin.Math.Soc.41(1923),p.63. "On the Theory of Graduation", Proc.Roy.Soc.Edin.44(1924), p.77.]

method can take its place in the regular practice of a life office. It will appear that to achieve this end considerable changes are required in the purely mathematical presentation of the theory: the principal novelties to which attention is drawn are the following:—

(1) The difference equation which is fundamental in the theory is solved by symbolic methods, and the Graduation Function of Professor Whittaker introduced* naturally as the coefficient of the general term of an expansion in powers of symbolic operators.

(2) The expansion in question, which proceeds by positive powers with increasing coefficients (graduation functions) is shown to be equivalent to an expansion in both positive and negative powers, with decreasing coefficients. The new general coefficient forms a much more tractable graduation function.

(3) It is shown that the theory implies the existence of auxiliary data beyond the ends of any given set, and that these can be found by a second and closely related symbolic expansion.

(4) A method of practical graduation is devised in which two series are required, one to annex data beyond the ends, the other to graduate the set augmented in this manner. By this means the whole of a given set of data may be graduated, without any being lost at the ends.

(5) A method is suggested for fixing upon a suitable value of ϵ, the parameter by which a balance is made between fidelity to the original data and regularity in successive differences.

[*The possibility of introducing the Graduation Function in this way was first noted in correspondence by R.Todhunter, 17/2/24.]

SYNOPSIS.

The following is a brief summary of the subject matter of the thesis, in the order of the sections in which it is treated:

§1. The meaning of the term "Graduation" is recalled, with a short account of the reasons for the necessity of the process, and of a compromise which has to be made between fidelity to the given data and regularity in successive differences.

§§2,3. The papers of Professor Whittaker on the subject are outlined, and the properties of the "Graduation Function" of the second paper described.

§§4,5. A temporary and more practicable solution of the problem is obtained by means of "graduation coefficients" which transform ungraduated into graduated values by a linear transformation.

§§6,7. The algebraic properties and mode of calculation of these coefficients are described.

§§8,9. Tables of the Graduation Function and its third differences are exhibited; also tables of the graduation coefficients, computed for selected values of n, the number of equidistant data to be graduated, and ϵ, the parameter by which more or less fidelity can be ensured.

§10. The foregoing tables are applied to some small sets of data, in order to obtain numerical material for examination.

§§11,12. The tables of graduation coefficients reveal certain remarkable properties, which suggest the introduction of convergent infinite graduating series. One of the latter is applied to an example already worked out in §10, and is found to give the same results with a greatly enhanced economy of apparatus.

§§13. The limiting values toward which the graduating coefficients in the tables were evidently tending as the number of data increased are found to be the successive coefficients of the terms of a convergent (Laurent) series, equivalent mathematically to the divergent (Taylor) series of which the Graduation Function was the general coefficient. The Graduation Function is henceforth discarded in favour of the new function. Also a second series is found by which auxiliary implied data may be annexed to the ends of a given set.

§§14,15,16,17. The calculation of the coefficients of the extrapolating and graduating series for selected values of ϵ likely to be of use in practice is described and performed, the results being entered in tables.

§§18,19. The practical application of the two series is described in detail with reference to the graduation of extended mortality tables, and the results are examined and discussed.

§20. General review and conclusions.

§1. THE MEANING OF "GRADUATION".

The meaning of the term "Graduation" and the necessity for such a process have been concisely described in the introductory section of the first paper of Professor Whittaker on the subject, Proc.Edin.Math.Soc. xli, p.63, and also in Whittaker and Robinson, "Calculus of Observations", §141, Ch. XI, p. 285, but for completeness a brief account may be given here.

When a set of numerical results derived from experimental or statistical observation is entered in a table corresponding to given values of some independent variable (such as the time) and the successive finite differences are formed in the usual manner, it is almost always found that the latter are highly irregular, in consequence of errors of observation and local peculiarities of the field from which the data have been gathered In general these irregularities are so marked as to defeat the object of such a table, namely, the computation of interpolated values, differential coefficients and integrals of the function by the methods of the Calculus of Finite Differences. Since therefore the data as they stand cannot be made the basis of successful prediction, a process of "smoothing" has to be performed upon them, and there arises inevitably the question, how far is fidelity to the original values to be sacrificed in order that the desired degree of regularity may be attained?

A familiar example of smoothing of a crude kind is the drawing of a "freehand" curve which shall pass as near as possible to points on a graph representing experimental results. Where the data are drawn from observations of large aggregates over extended periods of time so arbitrary a method will not suffice, and the problem of graduation must be subjected to analysis and treated as a part of applied mathematics.

As is only natural in a subject which arose first as a matter of practical necessity and not from theoretical considerations, this analysis has not been carried far enough, and the majority of the extant methods of graduation, efficient as they are in removing irregularities and often admirably ingenious, do not rest on well defined logical foundations. In most of them lie tacit assumptions and arbitrary elements, not always easy to disentangle and exhibit separately as postulates. [The commonest assumption, which is of course founded on good experience, is that if graduated values be derived from a sufficient number of ungraduated values, the errors in the latter, some being in excess and some in defect, will largely neutralize one another.] This insecurity of foundation could not be regarded with equanimity from the mathematical standpoint, and it became desirable to review the hypotheses upon which the whole subject was based. Such a review was instituted in the two papers about to be described.

§2. A THEORY OF GRADUATION BASED ON PROBABILITY.

In the papers of Professor Whittaker the problem of Graduation is regarded as belonging essentially to the mathematical theory of Probability, and a solution is developed in accordance with the principles of that theory. The important ideas are these:

If $u_1, u_2, u_3, \ldots, u_n$ are n numerical values of data requiring to be graduated, $u'_1, u'_2, u'_3, \ldots, u'_n$ the corresponding graduated values, two functions dominate the problem,

(i) the sum of the squares of the third differences of the graduated values,

$$S \equiv (u'_4 - 3u'_3 + 3u'_2 - u'_1)^2 + (u'_5 - 3u'_4 + 3u'_3 - u'_2)^2 + \ldots,$$

(ii) the sum of the squares of the "weighted" differences between corresponding graduated and ungraduated values,

$$F \equiv h_1^2(u'_1 - u_1)^2 + h_2^2(u'_2 - u_2)^2 + \ldots\ldots,$$

$h_1, h_2, \ldots\ldots, h_n$ being constants measuring the precision with which the respective observations have been made.

S provides a gauge of the irregularity of the graduated values, F provides a gauge of the fidelity of the graduated to the ungraduated values.

By means of the fundamental theorem of Inductive Probability the condition is arrived at that the most probable values $u'_1, u'_2, \ldots\ldots, u'_n$ of the graduated data are those which make

$$\lambda^2 S + F$$

a minimum, where λ is a constant.

The analytical formulation of the conditions for the required minimum leads to a set of linear equations between the u''s, their third differences and the u's; if the assumption is made that

$$h_1^2 = h_2^2 = \ldots\ldots\ldots = h_n^2 = \epsilon\lambda^2 ,$$

it is found that all the equations reduce to the type

$$\epsilon u'_x - \Delta^6 u'_{x-3} = \epsilon u_x ,$$

subject to the six terminal conditions

$$\Delta^3 u'_{-2} = 0,\ \Delta^3 u'_{-1} = 0,\ \Delta^3 u'_0 = 0,\ \Delta^3 u'_{n-2} = 0,\ \Delta^3 u'_{n-1} = 0,\ \Delta^3 u'_n = 0.$$

[The above linear difference equation of the sixth order is a central difference equation and may be written

$$\epsilon(u'_x - u_x) = \delta^6 u'_x .$$

That it ought to be a central difference equation follows from the remark that the same results should be obtained by graduating the data in the reversed order $u_n, u_{n-1}, \ldots\ldots, u_3, u_2, u_1$.]

In the first paper (Proc. Edin.Math.Soc.xli, p.63) this difference equation is solved by assuming an expansion of u'_x according to ascending powers of ϵ and neglecting squares and higher powers, the method being one of successive approximation. In the second paper (Proc. Roy. Soc. Edin. XLIV, p. 77) this method, which is applicable only when ϵ is a very small fraction, is replaced by a treatment superior in mathematical elegance in which is introduced an auxiliary function called the "Graduation Function".

§3. THE GRADUATION FUNCTION.

The Graduation Function, G_x, is defined by the series

$$G_x = (x+2)_5 + \epsilon(x+5)_{11} + \epsilon^2(x+8)_{17} + \ldots + \epsilon^r(x+3r+2)_{6r+5} + \ldots,$$

where $(m)_r$ denotes the binomial coefficient $m(m-1)(m-2)..(m-r+1)\div r!$. In the application x will always be a positive integer, so that G_x will be a terminating expression, a polynomial in ϵ .

The properties of G_x which make it useful for the present purpose are the following:—

(i) It is the coefficient of z^{-1} in the expansion in ascending powers of z^{-1} of

$$\frac{z^{x+2}}{(z-1)^6 - \epsilon z^3} ;$$

(ii) It satisfies the homogeneous (without second member) linear difference equation

$$\Delta^6 G_{x-3} - \epsilon G_x = 0, \text{ or } \delta^6 G_x - \epsilon G_x = 0,$$

with the terminal conditions

$$G_r = 0 \quad (r = -2,-1,0,1,2) \text{ and } G_3 = 1.$$

With the aid of G_x as an auxiliary function a solution of the problem of graduation is obtained in the form

$$v'_x = \epsilon(aG_x + bG_{x+1} + cG_{x+2} - p_x),$$

where

$$v'_x = \Delta^3 u'_x , \quad v_x = \Delta^3 u_x ,$$

$$p_x = v_1 G_{x-1} + v_2 G_{x-2} + \ldots\ldots + v_{x-3} G_3 ,$$

and a,b,c are constants (introduced in order to satisfy the terminal conditions) determined by the three linear equations

$$\begin{cases} aG_{n-2} + bG_{n-1} + cG_n = p_{n-2} , \\ aG_{n-1} + bG_n + cG_{n+1} = p_{n-1} , \\ aG_n + bG_{n+1} + cG_{n+2} = p_n . \end{cases}$$

The solution is then theoretically complete, for, the third differences of the graduated values being found, the graduated values themselves may be obtained by building back the difference table by summation.

§4. PRACTICAL DIFFICULTIES.

If G_x were fairly small over the range of values of x required in practice, the above method of solution could be applied as it stands. But it is often necessary to graduate from 50 to 100 data. Now G_x increases in what tends to become a geometrical progression, and even for small values of ϵ the values of G_x which enter into the calculation are very large. Since for a good graduation we shall be expecting quite small numbers for the graduated third differences, we shall find in the course of the work that huge numbers enter, annihilate each other with opposite signs and vanish, leaving insignificant residues. Not only so, but in order to obtain these residues accurately it will be necessary to calculate the constants a,b,c to at least the same number of digits as occurs before the decimal point in the largest G_x used. When to this is added the fact that it may be difficult to assign in advance a suitable value of ϵ, it will be seen that the obstacles in the way of practical solution on the above lines are insuperable.

Our first step was to inspect the various terms entering into the solution. Every v' will certainly depend on all the v's, so that we must reject the possibility of cutting down the latter in any way. Again, a solution of the difference equation by means of symbolic operators seems at first sight to show that the G_x's must enter into the problem as coefficients in an expansion of operators. It is thus natural, by exhaustion, to suppose that the constants a,b,c, special to each possible set of data, are the real cause of the trouble; that if they can be eliminated the part which is peculiar to each special problem will be eliminated with them; and that the remaining part, constant for all problems of graduation, ought surely to be calculable once and for all, however laborious the process of calculation.

The succeeding sections record the progress of these investigations, which force us in the end to modify the conclusion of the preceding paragraph and to abandon G_x in favour of a more tractable function.

§5. SOLUTION BY SYMBOLIC OPERATORS.

A treatment of the problem by symbolic methods (following the suggestion of R.Todhunter) is interesting, in that it shows clearly the manner in which the G_x's enter into the solution.

The problem is to determine u'_x by the condition that

$$\lambda^2 S + F \equiv \lambda^2 \sum_{r=1}^{n-3} (\Delta^3 u'_r)^2 + \sum_{r=1}^{n} (u'_r - u_r)^2$$

shall be a minimum. ....(1)

Any solution obtained will take account of unrestricted values of x, which may be outside the range 1 to n. This means that in order to preserve the minimum condition just given, we must have

$$\left.\begin{aligned} \Delta^3 u'_r &= 0 \\ u'_r &= u_r \end{aligned}\right\} \; (r=0,-1,-2,\ldots : \; r = n-2, n-1, n, \ldots) \qquad \ldots(2)$$

The ordinary conditions for a minimum then give a set of equations typified by

$$h_x^2(u'_x - u_x) = -\lambda^2(\Delta^3 u'_{x-3} - 3\Delta^3 u'_{x-2} + 3\Delta^3 u'_{x-1} - \Delta^3 u'_x),$$

i.e. by $h_x^2(u'_x - u_x) + \lambda^2 \Delta^6 u'_{x-3} = 0$.

Assuming provisionally $h_1^2 = h_2^2 = \ldots = h_n^2 = \epsilon\lambda^2$, we have to solve

$$\epsilon u'_x - \Delta^6 u'_{x-3} = \epsilon u_x ,$$

subject to the terminal conditions (2); i.e. writing $\Delta^3 u'_x = v'_x$, $\Delta^3 u_x = v_x$, we have to solve

$$\epsilon v'_x - \Delta^6 v'_{x-3} = \epsilon v_x ,$$

$$\text{with } \begin{cases} v'_r = 0 & (r = 0,-1,-2,\ldots;r = n-2,n-1,n,\ldots\ldots) \\ v_r = 0 & (r = -3,-4,-5,\ldots;r = n+1,\ n+2,\ \ldots\ldots). \end{cases}$$

Now $\quad v'_x = -[(1-E^{-1})^6 - \epsilon E^{-3}]^{-1} \epsilon v_{x-3}$, symbolically,

$$= -\epsilon[\sum_{r=0}^{\infty} G_r E^{-r}]\, v_x \text{ ; by the property of } G_x,\ \S 3,(i)$$

$$= -\epsilon \sum_{r=3}^{x+2} G_r v_{x-r} \quad (\text{since } v_{-3} = v_{-4} = \ldots = 0, G_2 = G_1 = G_0 = 0)$$

Written out more fully, these equations are

$$0 = v_{-2} G_{n+2} + v_{-1} G_{n+1} + \ldots\ldots\ldots\ldots + v_{n-3} G_3 \ldots\ldots (i)$$

$$0 = v_{-2} G_{n+1} + v_{-1} G_n + \ldots\ldots\ldots + v_{n-4} G_3 \ldots\ldots (ii)$$

$$0 = v_{-2} G_n + v_{-1} G_{n-1} + \ldots + v_{n-5} G_3 \ldots\ldots (iii)$$

$$\frac{v'_{n-3}}{\epsilon} = v_{-2} G_{n-1} + v_{-1} G_{n-2} + \ldots \ldots\ldots (iv)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{v'_1}{\epsilon} = v_{-2} G_3 . \ldots\ldots\ldots$$

[The v_0, v_{-1}, v_{-2} here are the a,b,c of the original paper with a change of sign.]

From equations (i),(ii),(iii),(iv) we can eliminate v_{-2}, v_{-1}, v_0 and solve for v'_{n-3}, obtaining

$$v'_{n-3} = \frac{\epsilon}{\begin{vmatrix} G_n & G_{n+1} & G_{n+2} \\ G_{n-1} & G_n & G_{n+1} \\ G_{n-2} & G_{n-1} & G_n \end{vmatrix}} \left\{ \sum_{r=1}^{n-3} v_r \begin{vmatrix} G_{n-r} & G_n & G_{n+1} & G_{n+2} \\ G_{n-r-1} & G_{n-1} & G_n & G_{n+1} \\ G_{n-r-2} & G_{n-2} & G_{n-1} & G_n \\ G_{n-r-3} & G_{n-3} & G_{n-2} & G_{n-1} \end{vmatrix} \right\}.$$

Similarly from equations (i),(ii),(iii),(v) we can solve for v'_{n-4} ; solving in general for v'_s , we obtain

$$v'_s = \epsilon \sum_{r=1}^{n-3} v_r \begin{vmatrix} G_{n-r} & G_n & G_{n+1} & G_{n+2} \\ G_{n-r-1} & G_{n-1} & G_n & G_{n+1} \\ G_{n-r-2} & G_{n-2} & G_{n-1} & G_n \\ G_{s-r} & G_s & G_{s+1} & G_{s+2} \end{vmatrix} \div \begin{vmatrix} G_n & G_{n+1} & G_{n+2} \\ G_{n-1} & G_n & G_{n+1} \\ G_{n-2} & G_{n-1} & G_n \end{vmatrix}.$$

The quotients of determinants form a set of graduating coefficients by which v'_x may be expressed linearly in terms of $v_1, v_2, \ldots, v_n$ for given values of n and ϵ.

But we need not stop short at third differences, for we can express u'_x in terms of $u_1, u_2, \ldots, u_n$ by means of a similar set of transforming coefficients. In fact, typifying the equations just obtained by

$$v'_s = \epsilon \sum_{r=1}^{n-3} v_r c_{r,s},$$

we have

$$u'_s - u_s = \epsilon \Delta_s^3 v'_{s-3} \quad \text{(by the original difference eqn.)}$$

$$= \Delta_s^3 \sum_{r=1}^{n-3} v_r c_{r,s-3}$$

$$= \Delta_s^3 \sum_{r=1}^{n-3} (u_{r+3} - 3u_{r+2} + 3u_{r+1} - u_r) c_{r,s-3}$$

$$= \sum_{r=1}^{n} u_r \Delta_r^3 \Delta_s^3 c_{r,s-3}.$$

Hence $\quad u'_s = \sum_{r=1}^{n} u_r g_{r,s}$, where

$$g_{r,s} = \begin{vmatrix} \Delta^3 G_{n-r} & G_n & G_{n+1} & G_{n+2} \\ \Delta^3 G_{n-r-1} & G_{n-1} & G_n & G_{n+1} \\ \Delta^3 G_{n-r-2} & G_{n-2} & G_{n-1} & G_n \\ \Delta^6 G_{s-r-3} & \Delta^3 G_{s-3} & \Delta^3 G_{s-2} & \Delta^3 G_{s-1} \end{vmatrix} \div \begin{vmatrix} G_n & G_{n+1} & G_{n+2} \\ G_{n-1} & G_n & G_{n+1} \\ G_{n-2} & G_{n-1} & G_n \end{vmatrix},$$

with unity added to this quotient when r = s, i.e. in the principal diagonal of the matrix $\|g_{r,s}\|$ of the linear transformation.

We have thus arrived at a set of coefficients by which the ungraduated values may be linearly transformed into the graduated, n and ϵ being supposed fixed. We shall refer to these in subsequent sections as the "graduation coefficients".

§6. ALGEBRAIC PROPERTIES OF THE GRADUATION COEFFICIENTS.

The graduation coefficients were evaluated for certain values of n and ϵ, to provide numerical material for inspection, with a view to conjecturing theorems by induction. Their evaluation is simplified considerably by their algebraic properties.

In the first place, recalling the remark that the same result must be obtained if any set of data be graduated in the reverse order, we must obviously have

$$g_{r,s} = g_{n-r+1,n-s+1}.$$

Again, from the determinantal expression for $g_{r,s}$, since the interchange of rows with columns makes no difference, we see that

$$g_{r,s} = g_{n-s+1,n-r+1}.$$

Combining this result with the previous one, we must have

$$g_{r,s} = g_{s,r} = g_{n-r+1,n-s+1} = g_{n-s+1,n-r+1}.$$

In other words, the matrix of the coefficients $\|g_{r,s}\|$ is both axisymmetric and centrosymmetric, or, what is the same thing, is symmetrical with regard to each diagonal.

Further, if we take in turn as ungraduated data the three sets of numbers

$$1,\ 1,\ 1,\ 1,\ \ldots\ldots\ldots\ldots,1,$$
$$1,\ 2,\ 3,\ 4,\ \ldots\ldots\ldots\ldots,n,$$
$$1^2,\ 2^2,\ 3^2,\ 4^2,\ \ldots\ldots\ldots\ldots,n^2\ ,$$

and graduate them in turn, it is obvious that we must merely reproduce the same sets again, since their third differences are zero.

Hence in any row (or column) of the matrix of the coefficients, such as the r^{th}, the moments of order 0, 1, 2 will be 1, r, r^2 respectively.

The two kinds of symmetry present are extremely valuable in that they reduce the number of coefficients to be calculated for any chosen n to almost one quarter of the whole number (e.g. if $n = 2m$ there are $4m^2$ coefficients in the matrix, but only $m.\overline{m+1}$ of them need to be calculated), while the properties of the moments not only serve as a useful check on the accuracy of the work, but may be used to locate any single appreciable error committed. For if an error k has been committed in the r^{th} member of the s^{th} row, say, the three moments will be in error by k, kr and kr^2 respectively. Hence if we find that the moments of a row are in error by numbers that are nearly in geometrical progression, with common ratio r, we shall look for the error either in the r^{th} member of the row or in one very near the r^{th}.

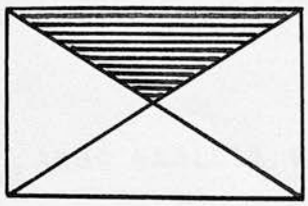
§7. CALCULATION OF THE GRADUATION COEFFICIENTS.

In order to see what results would be given by the theory, the writer calculated the graduation coefficients for certain selected values of n and ϵ. It was surmised that they would prove to be all numerically less than unity, and this proved to be the case. In spite of the help given by the double symmetry and the checks provided by the moment tests we must admit that the computation became tedious and laborious as n increased. We adopted a special routine, somewhat as follows:—

G_x and $\Delta^3 G_x$ were first tabulated to ten significant digits for $x = 0, 1, 2, \ldots, 32$, $\epsilon = 0$, $\cdot 01$, $\cdot 1$, 1 and 10. The value 10 of ϵ was taken as far as $x = 22$, not so much for any practical merit as for the interest of observing the change from regularity to fidelity as ϵ increases.

The invariable denominator of $g_{r,s}$, a persymmetric determinant of the third order, was then evaluated for each set of tables required, together with its six first minors.

In view of the double symmetry exhibited by the matrix, it was decided to calculate $g_{r,s}$ for $s = 1, 2, 3, \ldots, \frac{n}{2}$, $s \leqslant r \leqslant n-s+1$, i.e. to calculate the coefficients in the shaded part of the square matrix shown below, and to obtain the rest by reflexion about both diagonals.



The special advantage of this selection of r and s was that the lower left-hand corner element of the determinant in the numerator, $\Delta^6 G_{s-r-3}$, i.e. G_{s-r}, is zero for $s < r$. When $s = r$ $\Delta^6 G_{s-r-3}$ is unity, but then, as we have seen in §5, we must add unity to the quotient in the case of the principal diagonal elements. Hence in every case we have a quotient of the form

$$\begin{vmatrix} A & G_n & G_{n+1} & G_{n+2} \\ B & G_{n-1} & G_n & G_{n+1} \\ C & G_{n-2} & G_{n-1} & G_n \\ O & c & b & a \end{vmatrix} \div \begin{vmatrix} G_n & G_{n+1} & G_{n+2} \\ G_{n-1} & G_n & G_{n+1} \\ G_{n-2} & G_{n-1} & G_n \end{vmatrix},$$

where A,B,C,a,b,c are third differences of the G_x's, i.e. we have a bordered persymmetric determinant divided by the persymmetric determinant itself. Now the numerator may be expanded in terms of the border and the six first minors, and if we denote the quotients of those six minors when divided by the persymmetric determinant by $q_1, q_2, \ldots q_6$, the graduation coefficients assume the form

$$Aaq_1 + Bbq_2 + Ccq_3 - (Ab + Ba)q_4 + (Ac + Ca)q_5 - (Bc + Cb)q_6.$$

For each table we calculated the six q's, A,B,C,a,b,c were given by the table of third differences of G_x, and the rest of the work was performed with the aid of a calculating machine, supplemented by special arithmetical artifices too complicated to describe here.

The following pages exhibit the results of the calculations detailed in this section.

§8. TABLES OF THE GRADUATION FUNCTION AND THIRD DIFFERENCES.

The tables of G_x and $\Delta^3 G_x$ given below need very little explanation. The values are given to ten significant digits. In default of a different type, a different colour (red) is used to distinguish the prefixes or characteristics, which give the scale of the number. For example,

123·4567	would be represented by	3·1234567,
·1234567	....	0·1234567,
·001234567	....	$\bar{2}$·1234567,

and so on. Thus the prefix in red indicates the power of ten by which the proper fraction in blue, lying always between 1 and ·1, has to be multiplied. (1 is represented by 1·1.) This notation proved of the utmost service in all work done by machine.

The calculation of G_x and $\Delta^3 G_x$ for $\epsilon = 10$ was discontinued after the values x = 22 and x = 19 respectively. The graduated results obtained with this value of ϵ were interesting but, as might have been anticipated, of little practical use.

§9. TABLES OF THE GRADUATION COEFFICIENTS.

The tables of $g_{r,s}$ explain themselves. In view of the symmetry which leaves the table unaltered when inverted, only the upper half of each complete table has been given, the remainder being indicated by numbers in the margin in a manner exactly analogous to that by which sines and cosines (or tangents and cotangents) are often given by a single table.

The moment tests have been applied to every row of the tables. In a few cases the restriction to four places of decimals and the consequent forcing of the last place have caused a slight error in these moments, but this could hardly be avoided.

It is interesting to observe the changes which the increase of ϵ brings about in the matrices of the graduation coefficients. As one would expect, the increased fidelity is represented by a growing concentration of the larger coefficients in the main diagonal. Perfect fidelity would of course be represented when ϵ is infinite by a matrix consisting of units in the principal diagonal and zeros everywhere else, the "unit" matrix.

The tables show some other peculiarities which pure theory unaided by the results of calculation could hardly have predicted. Thus it appears that in the heart of each table $g_{r,s}$ tends more and more to become equal to $g_{r+1,s+1}$, and does this more notably if ϵ be large. This is exemplified in a striking manner in the table, n = 20, ϵ = 10; e.g. the main diagonal is ·9552, ·6821, ·5887, ·5809, ·5783, ·5770, ·5770, ·5770, ·5770, ·5770, and then the same numbers in the reversed order. It is evident that $g_{r,r}$ attains a limit which is approximately ·5770. What this limit actually is will appear in a later section.

A second feature which appears significant is that as n increases less and less change is produced in the large and important coefficients adjacent to the main diagonal and in it. Thus these coefficients tend to become independent of n. For instance, compare the small (left-hand top corner) sections of the tables given below.

n = 10: ϵ = ·1.

	1	2	3
1	·7487	·3329	·0763
2	·3329	·3097	·2336
3	·0763	·2336	·2883

n = 20: ϵ = ·1.

	1	2	3
1	·7477	·3333	·0778
2	·3333	·3074	·2306
3	·0778	·2306	·2835

n = 10: ϵ = 1.

	1	2	3
1	·8712	·2398	−·0329
2	·2398	·4250	·3008
3	−·0329	·3008	·4167

n = 20: ϵ = 1.

	1	2	3
1	·8712	·2398	−·0328
2	·2398	·4249	·3008
3	−·0328	·3008	·4165

n = 10: ϵ = 10.

	1	2	3
1	·9552	·1106	−·0647
2	·1106	·6821	·2703
3	−·0647	·2703	·5887

n = 20: ϵ = 10.

	1	2	3
1	·9552	·1106	−·0647
2	·1106	·6821	·2703
3	−·0647	·2703	·5887

[NOTE. The tables which follow were calculated in full for n = 10, 20. It was the original intention of the author to calculate them in full for n = 30, 40, 50 as well, but when part of the tables for n = 30 had been calculated the features above referred to suggested a détour by which such labour was rendered unnecessary. It will be seen later that these tables are really superseded, but we shall still exhibit them , not merely because they represent the result of arduous calculations but because they will illustrate important observations.]

TABLE OF G_x (to 10 significant digits).

x	ϵ =0.	ϵ =·01.	ϵ = ·1	ϵ = 1.	ϵ = 10.
3	1·1	1·1	1·1	1·1	1·1
4	1·6	1·6	1·6	1·6	1·6
5	2·21	2·21	2·21	2·21	2·21
6	2·56	2·5601	2·561	2·57	2·66
7	3·126	3·12612	3·1272	3·138	3·246
8	3·252	3·25278	3·2598	3·33	4·1032
9	3·462	3·4656401	3·49841	3·827	4·4202
10	3·792	3·8056518	3·92868	4·2175	5·16242
11	4·1287	4·13306971	4·172251	4·5826	5·62067
12	4·2002	4·2125874001	4·3251001	5·15519	6·240762
13	4·3003	4·3321838524	4·6245274	5·40836	6·943743
14	4·4368	4·51264537	5·1218984	6·106584	7·3693588
15	4·6188	4·78776973	5·239960701	6·277696	8·14390488
16	4·8568	5·1212978795	5·47318193	6·724968	8·55996128
17	5·11628	5·1879057578	5·930852765	7·189738	9·218153448
18	5·15504	5·2933771002	6·182432816	7·4972113	9·850711284
19	5·20349	5·4615610471	6·3563661424	8·13029534	10·3317210289
20	5·26334	5·7306330951	6·6947323573	8·34125561	11·1292990333
21	5·33649	6·1161282585	7·135343555	8·89336141	11·5039188875
22	5·42504	6·1849515795	7·2637288631	9·233831262	12·1964113465
23	5·5313	6·2946765364	7·5142338393	9·612074526	
24	5·6578	6·4691612544	8·1003353414	10·1602358863	
25	5·8073	6·7460004608	8·1958612517	10·4195173507	

TABLE OF G_x (continued)

x	$\epsilon = 0.$	$\epsilon = \cdot01.$	$\epsilon = \cdot1.$	$\epsilon = 1.$
26	5·9828	7·1184493991	8·382412921	11·109836455
27	6·118755	7·1878306909	8·746651316	11·2875634005
28	6·142506	7·2975589627	9·1457662536	11·7528523441
29	6·169911	7·4710942974	9·284537508	12·1970973372
30	6·201376	7·7456166322	9·5553642238	12·5160026481
31	6·237336	8·1180076162	10·1083909075	13·1350907351
32	6·278256	8·1867954685	10·2115450694	13·3536724346

TABLE OF $\Delta^3 G_x$ (to 10 significant digits).

x	ϵ = 0.	ϵ = ·01.	ϵ = ·1.	ϵ = 1.	ϵ = 10.
0	1·1	1·1	1·1	1·1	1·1
1	1·3	1·3	1·3	1·3	1·3
2	1·6	1·6	1·6	1·6	1·6
3	2·1	2·1001	2·101	2·11	2·2
4	2·15	2·1509	2·159	2·24	3·105
5	2·21	2·2145	2·255	2·66	3·471
6	2·28	2·296501	2·4451	3·194	4·1778
7	2·36	2·409515	2·8565	3·546	4·6486
8	2·45	2·57882	3·1749	4·1452	5·24915
9	2·55	2·85098001	3·362101	4·3739	5·99085
10	2·66	3·130656021	3·740121	4·9582	6·391416
11	2·78	3·207863031	4·1481511	5·24807	7·1522578
12	2·91	3·337977771	4·29113711	5·64933	7·5900191
13	3·105	3·5542186263	4·56542287	6·170796	8·22961685
14	3·12	3·9078501338	5·109290678	6·44898	8·8964294
15	3·136	4·1477649267	5·2113549541	7·1177181	9·349848836
16	3·153	4·2384913904	5·4100533083	7·3080367	10·1363540653
17	3·171	4·3817549804	5·7984710171	7·8055918	10·5312252871
18	3·19	4·6068934068	6·1559040897	8·21075947	11·2070309833
19	3·21	4·9600627741	6·3048129089	8·55169988	11·8070817992
20	3·231	5·1514326392	6·5960467952	9·144463602	
21	3·253	5·2385812519	7·1164949303	9·37829293	

TABLE OF $\Delta^3 G_x$ (continued)

x	ϵ = 0.	ϵ = ·01.	ϵ = .1.	ϵ = 1.
22	3·276	5·3759472735	7·2275249297	9·990489234
23	3·3	5·5929983575	7·4441180615	10·259312704
24	3·325	5·9366506295	7·8666096671	10·6788565211
25	3·351	6·1481504136	8·1690860998	11·1777197725
26	3·378	6·2346008275	8·3299284976	11·4652700867
27	3·406	6·3717993738	8·6438532916	12·1218099995
28	3·435	6·5895019488	9·1256626736	12·3189061842
29	3·465	6·9348179822	9·2452786337	12·8349128999

TABLE OF GRADUATION COEFFICIENTS (to four places).

$n = 10: \epsilon = 0.$

	1	2	3	4	5	6	7	8	9	10	
1	·6182	·3818	·1909	·0455	-·0545	-·1091	-·1182	-·0818	·0000	·1273	10
2	·3818	·2788	·1909	·1182	·0606	·0182	-·0091	-·0212	-·0182	·0000	9
3	·1909	·1909	·1833	·1682	·1455	·1152	·0773	·0318	-·0212	-·0818	8
4	·0455	·1182	·1682	·1955	·2000	·1818	·1409	·0773	-·0091	-·1182	7
5	-·0545	·0606	·1455	·2000	·2242	·2182	·1818	·1152	·0182	-·1091	6
	10	9	8	7	6	5	4	3	2	1	

$n = 10: \epsilon = {\cdot}01.$

	1	2	3	4	5	6	7	8	9	10	
1	·6452	·3729	·1674	·0250	-·0611	-·0993	-·0975	-·0615	·0057	·1030	10
2	·3729	·2831	·1987	·1232	·0609	·0145	-·0145	-·0257	-·0189	·0057	9
3	·1674	·1987	·2043	·1859	·1505	·1061	·0595	·0149	-·0257	-·0615	8
4	·0250	·1232	·1859	·2131	·2071	·1747	·1236	·0595	-·0145	-·0975	7
5	-·0611	·0609	·1505	·2071	·2294	·2173	·1747	·1061	·0145	-·0993	6
	10	9	8	7	6	5	4	3	2	1	

TABLE OF GRADUATION COEFFICIENTS (continued)

$n = 10 : \epsilon = \cdot 1.$

	1	2	3	4	5	6	7	8	9	10	
1	·7487	·3329	·0763	−·0461	−·0765	−·0582	−·0245	·0048	·0209	·0218	10
2	·3329	·3097	·2336	·1378	·0531	−·0045	−·0324	−·0348	−·0163	·0209	9
3	·0763	·2336	·2883	·2480	·1590	·0659	−·0034	−·0377	−·0348	·0048	8
4	−·0461	·1378	·2480	·2799	·2335	·1472	·0600	−·0034	−·0324	−·0245	7
5	−·0765	·0531	·1590	·2335	·2590	·2215	·1472	·0659	−·0045	−·0582	6
	10	9	8	7	6	5	4	3	2	1	

$n = 10 : \epsilon = 1.$

	1	2	3	4	5	6	7	8	9	10	
1	·8712	·2398	−·0329	−·0754	−·0344	·0040	·0169	·0126	·0035	−·0052	10
2	·2398	·4250	·3008	·1071	−·0120	−·0420	−·0260	−·0035	·0075	·0035	9
3	−·0329	·3008	·4167	·2819	·0987	−·0109	−·0387	−·0248	−·0035	·0126	8
4	−·0754	·1071	·2819	·3738	·2634	·1015	−·0044	−·0387	−·0260	·0169	7
5	−·0344	−·0120	·0987	·2634	·3668	·2650	·1015	−·0109	−·0420	·0040	6
	10	9	8	7	6	5	4	3	2	1	

$n = 10 : \epsilon = 10.$

	1	2	3	4	5	6	7	8	9	10	
1	·9552	·1106	−·0647	−·0187	·0109	·0075	·0005	−·0011	−·0004	·0002	10
2	·1106	·6821	·2703	−·0186	−·0456	−·0075	·0064	·0031	−·0005	−·0004	9
3	−·0647	·2703	·5887	·2434	−·0029	−·0349	−·0069	·0049	·0031	−·0011	8
4	−·0187	−·0186	·2434	·5809	·2479	·0000	−·0349	−·0069	·0064	·0005	7
5	·0109	−·0456	−·0029	·2479	·5783	·2462	·0000	−·0349	−·0075	·0075	6
	10	9	8	7	6	5	4	3	2	1	

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TABLE OF GRADUATION COEFFICIENTS (continued)

n = 20: ϵ = 0.

n = 20: ϵ = 0.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
1	·3708	·2981	·2318	·1721	·1188	·0721	·0318	-·0019	-·0292	-·0500	-·0643	-·0721	-·0734	-·0682	-·0565	-·0383	-·0136	·0175	·0552	·0994	20
2	·2981	·2453	·1970	·1531	·1136	·0784	·0481	·0220	·0003	-·0169	-·0297	-·0380	-·0419	-·0413	-·0364	-·0270	-·0131	·0052	·0280	·0552	19
3	·2318	·1970	·1647	·1351	·1081	·0837	·0620	·0428	·0263	·0124	·0011	-·0075	-·0136	-·0170	-·0178	-·0160	-·0115	-·0045	·0052	·0175	18
4	·1721	·1531	·1351	·1181	·1022	·0873	·0734	·0606	·0488	·0380	·0282	·0194	·0117	·0050	-·0007	-·0053	-·0089	-·0115	-·0131	-·0136	17
5	·1188	·1136	·1081	·1022	·0960	·0894	·0825	·0753	·0677	·0598	·0515	·0429	·0339	·0246	·0150	·0050	-·0053	-·0160	-·0270	-·0383	16
6	·0721	·0784	·0837	·0873	·0894	·0901	·0892	·0869	·0831	·0778	·0711	·0628	·0531	·0419	·0292	·0150	-·0007	-·0178	-·0364	-·0565	15
7	·0318	·0481	·0620	·0734	·0825	·0892	·0935	·0955	·0950	·0921	·0868	·0792	·0691	·0567	·0419	·0246	·0050	-·0170	-·0413	-·0682	14
8	-·0019	·0220	·0428	·0606	·0753	·0869	·0955	·1009	·1033	·1026	·0989	·0920	·0821	·0691	·0531	·0339	·0117	-·0136	-·0419	-·0734	13
9	-·0292	·0003	·0263	·0488	·0677	·0831	·0950	·1033	·1081	·1094	·1071	·1014	·0920	·0792	·0628	·0429	·0194	-·0075	-·0380	-·0721	12
10	-·0500	-·0169	·0124	·0380	·0598	·0778	·0921	·1026	·1094	·1124	·1117	·1071	·0989	·0868	·0711	·0515	·0282	·0011	-·0297	-·0643	11
	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	

n = 20: ϵ = ·01.

n = 20: ϵ = ·01.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
1	·6066	·3603	·1767	·0519	-·0223	-·0570	-·0640	-·0544	-·0371	-·0187	-·0032	·0078	·0139	·0158	·0146	·0113	·0070	·0021	-·0030	-·0084	20
2	·3603	·2768	·1985	·1291	·0722	·0296	·0013	-·0145	-·0207	-·0201	-·0158	-·0099	-·0041	·0006	·0038	·0054	·0054	·0040	·0011	-·0030	19
3	·1767	·1985	·1974	·1752	·1391	·0977	·0583	·0256	·0020	-·0123	-·0187	-·0191	-·0159	-·0108	-·0055	-·0009	·0024	·0040	·0040	·0021	18
4	·0519	·1291	·1752	·1906	·1782	·1467	·1063	·0655	·0306	·0046	-·0119	-·0198	-·0212	-·0182	-·0131	-·0073	-·0019	·0024	·0054	·0070	17
5	-·0223	·0722	·1391	·1782	·1894	·1751	·1433	·1034	·0637	·0297	·0043	-·0118	-·0197	-·0212	-·0185	-·0134	-·0073	-·0009	·0054	·0113	16
6	-·0570	·0296	·0977	·1467	·1751	·1815	·1663	·1359	·0985	·0612	·0291	·0049	-·0108	-·0190	-·0210	-·0185	-·0131	-·0055	·0038	·0146	15
7	-·0640	·0013	·0583	·1063	·1433	·1663	·1716	·1580	·1304	·0957	·0605	·0297	·0058	-·0103	-·0190	-·0212	-·0182	-·0108	·0006	·0158	14
8	-·0544	-·0145	·0256	·0655	·1034	·1359	·1580	·1648	·1535	·1281	·0951	·0609	·0301	·0058	-·0108	-·0197	-·0212	-·0159	-·0041	·0139	13
9	-·0371	-·0207	·0020	·0306	·0637	·0985	·1304	·1535	·1618	·1520	·1277	·0952	·0609	·0297	·0049	-·0118	-·0198	-·0191	-·0099	·0078	12
10	-·0187	-·0201	-·0123	·0046	·0297	·0612	·0957	·1281	·1520	·1610	·1517	·1277	·0951	·0605	·0291	·0043	-·0119	-·0187	-·0158	-·0032	11
	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	

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TABLE OF GRADUATION COEFFICIENTS (continued)

$n = 20 : \epsilon = \cdot 1.$

$n = 20 : \epsilon = \cdot 1.$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
1	·7477	·3333	·0778	-·0442	-·0748	-·0579	-·0267	-·0007	·0134	·0165	.0129	·0071	·0020	-·0012	-·0024	-·0022	-·0014	-·0004	·0003	·0008	20
2	·3333	·3074	·2306	·1361	·0547	·0016	-·0226	-·0258	-·0184	-·0082	-·0005	·0036	·0045	·0035	·0018	·0003	-·0006	-·0009	-·0006	·0003	19
3	·0778	·2306	·2835	·2442	·1591	·0725	·0099	-·0224	-·0300	-·0235	-·0124	-·0027	·0030	·0049	·0042	·0024	·0007	-·0005	-·0009	-·0004	18
4	-·0442	·1361	·2442	·2757	·2311	·1490	·0678	·0097	-·0200	-·0271	-·0212	-·0111	-·0024	·0028	·0045	·0039	·0023	·0007	-·0006	-·0014	17
5	-·0748	·0547	·1591	·2311	·2536	·2139	·1411	·0676	·0137	-·0152	-·0233	-·0192	-·0106	-·0027	·0023	·0042	·0039	·0024	·0003	-·0022	16
6	-·0579	·0016	·0725	·1490	·2139	·2403	·2078	·1410	·0707	·0174	-·0123	-·0218	-·0189	-·0109	-·0030	·0023	·0045	·0042	·0018	-·0024	15
7	-·0267	-·0226	·0099	·0678	·1411	·2078	·2375	·2078	·1424	·0724	·0187	-·0117	-·0218	-·0190	-·0109	-·0027	·0028	·0049	·0035	-·0012	14
8	-·0007	-·0258	-·0224	·0097	·0676	·1410	·2078	·2375	·2078	·1424	·0724	·0186	-·0118	-·0218	-·0189	-·0106	-·0024	·0030	·0045	·0020	13
9	·0134	-·0184	-·0300	-·0200	·0137	·0707	·1424	·2078	·2368	·2070	·1418	·0721	·0186	-·0117	-·0218	-·0192	-·0111	-·0027	·0036	·0071	12
10	·0165	-·0082	-·0235	-·0271	-·0152	·0174	·0724	·1424	·2070	·2359	·2065	·1418	·0724	·0187	-·0123	-·0233	-·0212	-·0124	-·0005	·0129	11
	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	

$n = 20 : \epsilon = 1.$

$n = 20 : \epsilon = 1.$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
1	·8712	·2398	-·0328	-·0752	-·0343	·0039	·0162	·0116	·0036	-·0011	-·0020	-·0011	-·0002	·0003	·0003	·0001	·0000	·0000	·0000	·0000	20
2	·2398	·4249	·3008	·1073	-·0115	-·0415	-·0262	-·0054	·0050	·0056	·0026	·0001	-·0008	-·0006	-·0002	·0001	·0001	·0001	·0000	·0000	19
3	-·0328	·3008	·4165	·2816	·0986	-·0105	-·0374	-·0233	-·0045	·0047	·0051	·0023	·0000	-·0008	-·0006	-·0002	·0001	·0001	·0001	·0000	18
4	-·0752	·1073	·2816	·3726	·2616	·1008	-·0010	-·0306	-·0212	-·0051	·0035	·0045	·0022	·0002	-·0006	-·0005	-·0002	·0001	·0001	·0000	17
5	-·0343	-·0115	·0986	·2616	·3635	·2627	·1051	·0021	-·0297	-·0215	-·0057	·0032	·0044	·0023	·0003	-·0006	-·0005	-·0002	·0001	·0001	16
6	·0039	-·0415	-·0105	·1008	·2627	·3633	·2622	·1048	·0019	-·0296	-·0215	-·0056	·0032	·0044	·0023	·0003	-·0006	-·0006	-·0002	·0003	15
7	·0162	-·0262	-·0374	-·0010	·1051	·2622	·3613	·2607	·1044	·0021	-·0294	-·0213	-·0056	·0032	·0044	·0023	·0002	-·0008	-·0006	·0003	14
8	·0116	-·0054	-·0233	-·0306	·0021	·1048	·2607	·3603	·2604	·1045	·0023	-·0293	-·0213	-·0056	·0032	·0044	·0022	·0000	-·0008	-·0002	13
9	·0036	·0050	-·0045	-·0212	-·0297	·0019	·1044	·2604	·3602	·2604	·1045	·0023	-·0293	-·0213	-·0056	·0032	·0045	·0023	·0001	-·0011	12
10	-·0011	·0056	·0047	-·0051	-·0215	-·0296	·0021	·1045	·2604	·3602	·2604	·1045	·0023	-·0294	-·0215	-·0057	·0035	·0051	·0026	-·0020	11
	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	

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TABLE OF GRADUATION COEFFICIENTS (continued)

$n = 20;\ \epsilon = 10.$

$n = 20;\ \epsilon = 10.$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
1	·9552	·1106	-·0647	-0187	·0109	·0075	·0005	-·0011	-0004	·0001	·0001	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000	20
2	·1106	·6821	·2703	-·0186	-·0456	-·0075	·0063	·0032	-·0002	-·0006	-·0001	·0001	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000	19
3	-·0647	·2703	·5887	·2434	-·0029	-·0348	-·0069	·0048	·0026	-·0001	-·0005	-·0001	·0001	·0000	·0000	·0000	·0000	·0000	·0000	·0000	18
4	-·0187	-·0186	·2434	·5809	·2479	·0002	-·0346	-·0073	·0046	·0027	·0000	-·0005	-·0001	·0000	·0000	·0000	·0000	·0000	·0000	·0000	17
5	·0109	-·0456	-·0029	·2479	·5783	·2461	·0001	-·0343	-·0072	·0046	·0027	·0000	-·0005	-·0001	·0000	·0000	·0000	·0000	·0000	·0000	16
6	·0075	-0075	-·0348	·0002	·2461	·5770	·2461	·0003	-·0343	-·0072	·0046	·0027	·0000	-·0005	-·0001	·0000	·0000	·0000	·0000	·0000	15
7	·0005	·0063	-·0069	-·0346	·0001	·2461	·5770	·2461	·0003	-·0343	-·0072	·0046	·0027	·0000	-·0005	-·0001	·0000	·0000	·0000	·0000	14
8	-·0011	·0032	·0048	-·0073	-0343	·0003	·2461	·5770	·2461	·0003	-·0343	-·0072	·0046	·0027	·0000	-·0005	-·0001	·0001	·0000	·0000	13
9	-·0004	-·0002	·0026	·0046	-·0072	-·0343	·0003	·2461	·5770	·2461	·0003	-·0343	-·0072	·0046	·0027	·0000	-·0005	-·0001	·0001	·0000	12
10	·0001	-·0006	-0001	·0027	·0046	-·0072	-0343	·0003	·2461	·5770	·2461	·0003	-·0343	-·0072	·0046	·0027	·0000	-·0005	-0001	·0001	11
	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	

§10. EXAMPLES OF ACTUAL GRADUATION.

Examples will now be given of the application of the tables of graduation coefficients to the graduation of actual data derived from mortality statistics. The sets of data are of course small, not exceeding 20 in number. The truth is that as yet we have not found the ideal practical method which lies potential in the theory, and the best we can do is to use what we have so far obtained and examine the results.

No subtlety was involved in the method of computation, which simply consisted in writing the data vertically below each other on a long strip of paper at such intervals that they could be aligned with the elements of a column of graduation coefficients, and forming and summing the products $u_s g_{r,s}$ for each column in turn on a machine which performed the two operations at once.

The first set of data taken was the set of ten to which the original method of successive approximation had been applied. (Whittaker and Robinson, "Calculus of Observations", pp. 312-315). We elected to graduate this set because we suspected that the approximation method had given quite erroneous results for the value of ϵ, ·08, there taken. (loc.cit.) These doubts proved to be well founded.

The results of the graduation for the five different values of ϵ are tabulated on the next page.

Example from Whittaker and Robinson, "Calculus of Observations", pp. 312-315, graduated by the present method for ϵ = ·01, ·1, 1 and 10.

Data.	Succ.App.	By Graduation Coefficients.			
	ϵ = ·01.	ϵ = ·01.	ϵ = ·1.	ϵ = 1.	ϵ = 10.
1019	1219	1221	1144	1072	1037
Δ^3	Δ^3	Δ^3	Δ^3	Δ^3	Δ^3
1550	1386	1385	1410	1452	1497
551	4	2	12	56	181
1611	1525	1523	1590	1653	1668
-204	3	6	24	58	27
1753	1640	1637	1696	1731	1731
-120	8	6	31	65	94
1772	1734	1733	1752	1744	1713
941	9	5	29	44	171
1548	1815	1817	1789	1757	1708
-1271	3	7	18	-33	-345
2022	1892	1894	1836	1814	1887
591	7	3	17	46	165
1923	1968	1971	1911	1882	1905
550	1	1	12	70	342
1842	2050	2051	2031	2007	1927
2329	2139	2135	2208	2259	2295
M_0 = 17369	17368	17367	17367	17371	17368
M_1 =103518	103514	103510	103511	103528	103514
M_2 =754046	754028	753973	754013	754106	754016
S=3512320	230	160	3279	20766	334781
F= 0	258083	261516	182498	136196	62877

Of the graduated results, those given by $\epsilon = \cdot 1$ are very satisfactory from the point of view of smallness of third differences, and are sufficiently faithful to the original data. [The stages of the conflict between fidelity and smoothness in the different results may be followed by the changes in the respective values of S and F in the table.] The fact that the third differences for $\epsilon = \cdot 1$ were notably smaller than those obtained by successive approximation for $\epsilon = \cdot 08$ in the "Calculus of Observations", whereas they should have been a little greater, confirmed our suspicions. We therefore computed graduation coefficients for $\epsilon = \cdot 08$ and graduated the set again. It will be seen by the comparison below, in which the erroneous values are shown in red below the true values, that the error in the former is considerable. Hence we may fairly condemn the method of successive approximation as unreliable, since it leads to false results for the very values of ϵ which are likely to be of service.

$\epsilon = \cdot 08.$

Correct	1154,	1407,	1582,	1690,	1750,	1792,	1843,	1918,	2033,	2200.
Incorrect	1048,	1436,	1673,	1775,	1784,	1758,	1757,	1828,	2005,	2305.
Error	-106,	+29,	+91,	+85,	+34,	-34,	-86,	-90,	-28,	+105.

Had $\epsilon = \cdot 1$ been taken the error would have been greater.

$\epsilon = \cdot 1.$

Correct	1144,	1410,	1590,	1696,	1752,	1789,	1836,	1911,	2031,	2208.
Incorrect	999,	1451,	1715,	1814,	1799,	1741,	1718,	1788,	1992,	2352.
Error	-145,	+41,	+125,	+118,	+47,	-48,	-118,	-123,	-39,	+144.

A peculiar fact is that the three moments M_0, M_1, M_2 of each of the two sets of incorrect results given above are remarkably close to the corresponding moments of the original data. This shows that the test by moments of graduated results is corroborative evidence of accuracy of computation, but by no means conclusive evidence.

A second point of interest which we did not pause to investigate was that the errors induced by the approximation method were symmetrical, with a change of sign, about the central pair. (See the previous page .) This suggested that if an odd number of data were graduated, not, as here, an even number, by the approximation method, the middle graduated value would be correct, though the remainder were wrong.

As a second example part of a table showing rates of mortality of female annuitants* in the United Kingdom during the period 1900-1920 was taken and graduated. Since we had not yet considered how to cope with cases in which the modulus of precision varied greatly, we graduated the section of this table showing the rate of mortality between ages 60 and 79 inclusive: the "exposed to risk" for these ages did not depart excessively from a mean of about 12,000.

The graduated results for the five different values of ϵ are tabulated on the next page, together with their moments, their third differences and the corresponding values of F.

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* ["Report on the results of an investigation of the mortality experience of life annuitants during the period 1900-1920", by W.Palin Elderton and H.J.P.Oakley, J.I.A. LIV(1923),p.43-;p.83.]

Age	Ungrad. $q.10^5$	Graduated by Graduation Coefficients $\epsilon=0$.	$\epsilon=\cdot01$.	$\epsilon=\cdot1$.	$\epsilon=1$.	$\epsilon=10$.
	Δ^3	Δ^3	Δ^3	Δ^3	Δ^3	Δ^3
60	1417	1762	1537	1480	1453	1416
61	1721	1759	1681	1676	1708	1753
	-457	4	5	7	37	-7
62	2017	1794	1824	1848	1872	1927
	884	-5	0	15	86	293
63	1848	1871	1971	2003	1982	1931
	-541	4	6	7	22	-3
64	2098	1985	2122	2156	2124	2058
	486	-2	3	3	-41	-53
65	2226	2140	2283	2314	2320	2305
	-863	-1	7	-1	-65	-268
66	2718	2334	2457	2480	2529	2619
	585	3	11	19	41	141
67	2711	2566	2651	2653	2686	2732
	-36	-2	5	21	81	201
68	2790	2839	2876	2852	2832	2785
	287	-1	13	17	45	93
69	2919	3151	3137	3098	3048	2979
	-164	3	0	18	-44	-221
70	3385	3501	3447	3408	3379	3407
	-829	-3	2	-7	-38	-173
71	4024	3892	3806	3780	3781	3848
	1190	1	5	1	41	448
72	4007	4321	4216	4207	4216	4129
	171	-1	-4	-12	-43	-101
73	4524	4789	4682	4690	4725	4698
	-1921	4	0	-5	-79	-657
74	5746	5295	5200	5217	5265	5454
	1286	-2	1	11	132	549
75	5752	5843	5770	5783	5757	5740
	1257	-2	-1	15	95	566
76	5828	6431	6393	6399	6333	6105
	-1815	1	0	-22	-153	-708
77	7231	7057	7068	7080	7088	7115
	-197	3	-4	-17	-153	-518
78	8146	7722	7795	7804	7869	8062
79	8376	8429	8570	8554	8523	8428
M_0	79484	79481	79486	79482	79490	79491
M_1	1067862	1067836	1067899	1067845	1067914	1067936
M_2	16649716	16649608	16650215	16649451	16650484	16650506
F	0	1403451	1123928	1071494	621713	299695

As in the previous example of this section, the best results seem to be given by the value $\epsilon = \cdot 1$. This seems to indicate that this particular value of ϵ is likely to be of frequent use. Values of ϵ greater than unity, on the other hand, are extremely unlikely to be of practical service.

The table of results shows even better than in the previous example the stages of the conflict between fidelity and smoothness as ϵ increases.

[A check on the accuracy of the values given in the table is that their sixth differences should satisfy the fundamental central difference equation. Owing to the accumulation of error produced in a sixth difference column by small unavoidable deviations from perfect accuracy in the argument column due to the forcing of the last digit, this check cannot be applied satisfactorily when ϵ is small. It might also be thought that after finding seven consecutive graduated results we could use the difference equation successively to find the rest, but the rapid accumulation of error militates against this likewise.

The check by sixth differences can be used to demonstrate the accuracy of the results we have obtained for the large value $\epsilon = 10$. We ought to have $\delta^6 u'_r = 10(u'_r - u_r)$. The respective values of $(u'_r - u_r)$ and $\delta^6 u'_r$ given below show that this is very nearly the case.

$u'_r - u_r$	83	-40	79	-99	21	-5	60	22	-176	122	174	-292
$\delta^6 u'_r$	842	-411	789	-973	181	-38	568	211	-1743	1163	1769	-2951

$u'_r - u_r$	-12	277
$\delta^6 u'_r$	-102	2755

We shall have occasion to rely on the accuracy of this particular graduation in future developments.]

§11. A MODIFIED METHOD OF GRADUATION.

The tables of graduation coefficients have so far served their purpose well enough, and we have obtained by means of them some results which will be useful as a test of any modified method we may adopt. We are bound to consider whether any modification is feasible, for it is obvious that if we have to have a separate table of coefficients for every pair of values of n and ϵ likely to be required in practice, the tables are going to be bulky and voluminous to excess, and the calculations will be tedious and confusing, on account of the numerous different columns in each table. These are very serious disabilities, and it will a vast improvement if for each value of ϵ we can give a single table of coefficients which may be used for any value whatever of n, and further if we can reduce this single table to as few columns as possible. We shall now show how the properties of the matrices of graduation coefficients suggest a way of accomplishing this very desirable reduction.

It was observed in those matrices for n = 10, 20 and the few rows that were calculated for n = 30 that the large entries in and adjacent to the principal diagonals appeared to be tending to limits depending on ϵ but not on n, and that in the centre of the tables the values of $g_{r,s}$ and of $g_{r+1,s+1}$ were more and more nearly equal. The explanation of these properties was surmised to be as follows:—

The original terminal conditions which the solution of the fundamental central difference equation had to satisfy were

that $\quad \Delta^3 u'_r = 0$
$\quad u'_r = u_r \quad (r = 0,-1,-2,\ldots;\ r = n-2,n-1,n,n+1,\ldots.).$

Suppose now that by some method yet to be evolved we have found the implied ungraduated data lying beyond the termini of the original limited set in either direction, i.e. that we know the values of $u_0, u_{-1}, u_{-2},\ldots$ and also of $u_{n+1}, u_{n+2}, u_{n+3},\ldots$ Suppose further that we augment the original set by attaching as many as we please of these newly found data to their respective ends of the original sequence. Now consider the result of graduating the augmented set according to the difference equation. It is obvious, since the third differences of the graduated data will be all zero beyond the original termini, that $u_0, u_{-1}, u_{-2}\ldots$ and $u_{n+1}, u_{n+2}, u_{n+3},\ldots$ will be unchanged, while the original set in the middle $u_1, u_2, u_3,\ldots u_n$ will be replaced by exactly the same graduated values as we should have obtained had we graduated them <u>unaugmented</u>, namely by $u'_1, u'_2, u'_3,\ldots u'_n$. The sole change has been to increase the number of data requiring to be graduated.

This being so, let us take the further step of annexing in this manner to either end of the original sequence an <u>infinite</u> number of new "ungraduated" data. It is natural to conjecture, and we shall find later that it is true, that the coefficients $g_{r,s}$ tend to <u>limits</u> in the centre of the tables, and that we shall now have each graduated value u'_r given by an <u>infinite series</u> involving the ungraduated values linearly, of the form

$$u'_r = k_0 u_r + k_1(u_{r+1} + u_{r-1}) + k_2(u_{r+2} + u_{r-2}) + \ldots\ldots \text{ ad inf.},$$

where the k's now depend on ϵ alone. It now remains to hope fervently that for serviceable values of ϵ the succession of k's may decrease with such rapidity that only a limited small number of them will be needed to obtain results of sufficient accuracy.

[NOTE. The phenomena apparent in the tables now permit of simple explanation. The graduation ofan infinite number of data is really a linear transformation represented by an infinite matrix, symmetric about both diagonals, and obviously such that $g_{r,s} = g_{r+1,s+1}$. The values of the elements of this matrix will decrease as we recede from the principal diagonal. In our tables, by regarding only a small finite set of data in the middle of the sequence, and blotting out the rest by the condition that the graduated and ungraduated values shall coincide, we have as it were condensed this infinite matrix into a small equivalent finite matrix. The effects of this condensation have been most marked in the large elements near the boundary and in and adjacent to the principal diagonal, while those more symmetrically placed in the central portion of the matrix have suffered least.]

The coefficients k_r in the infinite series given above are placed symmetrically about a central coefficient k_0, and are the limits of the elements in the middle row (or column) of any matrix for a given value of ϵ. But it is equally apparent from our tables that the elements of all the other rows or columns, such as the first, second, and so on, have similarly been tending to limiting values. We shall now show that if we retain the first row in its limiting form as a sequence of coefficients which decrease indefinitely, $i_0, i_1, i_2, \ldots..$ say, we have material for a simple and effective method of graduation, which actually becomes the more reliable the larger n is.

For we shall have, since graduated and ungraduated data beyond the termini coincide,

$$u'_0 = u_0 = i_0 u_0 + i_1 u_1 + i_2 u_2 + \ldots\ldots,$$

and if this converges rapidly enough, u_0 is found as closely as we desire.

But then, having found u_0, we may repeat the process, for

$$u'_{-1} = u_{-1} = i_0 u_{-1} + i_1 u_0 + i_2 u_1 + \ldots\ldots,$$

whence u_{-1} may be determined. Similarly u_{-2}, u_{-3}, ... may be determined. In reality we need only find three u's beyond the ends, u_0, u_{-1}, u_{-2}, for by forming a difference table and using the fact that third differences outside the range of graduation are all zero we may quickly determine as many other u's as we require. If a check is desired, we may find four u's by the method given above, and the single third difference to which these give rise ought to be zero.

Exactly the same process applied to the reversed data from the other end will give as many of u_{n+1}, u_{n+2}, u_{n+3}, as we require.

Then six graduated data, three at each end, may be found immediately by using the original terminal conditions, i.e. by inserting three more zeros inwards in the third difference column and building back the table by summation. The remainder of the graduated values may be obtained by applying to the set of u's, augmented by annexing the extrapolated values, the graduating k-series.

§12. ILLUSTRATIVE EXAMPLE OF THE METHOD.

Postponing for the moment the full consideration of what functions the limiting coefficients i_r and k_r really are, we shall show by a worked example the application of the method. For this purpose we shall take the set of twenty data already used in the second example of §10, and shall choose $\epsilon = 10$. Of course we know beforehand from the results of §10 that we shall obtain graduated values of no practical use at all, but this value of ϵ serves very well to exhibit the method in a small compass and instructively. We shall assume that the limits of i_r and k_r are very nearly those shown in our table for $n = 20$, $\epsilon = 10$, i.e. we shall assume the values given below:

i_0	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9	i_{10}
·9552	·1106	−·0647	−·0187	·0109	·0075	·0005	−·0011	−·0004	·0001	·0001

k_0	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}
·5770	·2461	·0003	−·0343	−·0072	·0046	·0027	·0000	−·0005	−·0001	−·0001

The set of data we are going to graduate is:

u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}
1417	1721	2017	1848	2098	2226	2718	2711	2790	2919	3385

u_{12}	u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}
4024	4007	4524	5746	5752	5828	7231	8146	8376

First, $u_0' = u_0 = \ \cdot 9552\, u_0 + \cdot 1106(1417) - \cdot 0647(1721) - \ldots,$

i.e. $\cdot 0448\, u_0 = 41 \cdot 14$, so that $u_0 = 918$.

Next, $u'_{-1} = \cdot 9552\, u_{-1} + \cdot 1106(918) - \cdot 0647(1417) - \ldots\ldots,$

i.e. $\cdot 0448\, u_{-1} = 11\cdot 58$, so that $u_{-1} = 258$.

Similarly $\cdot 0448\, u_{-2} = \cdot 1106(258) - \cdot 0647(918) - \ldots\ldots$, so that $\cdot 0448\, u_{-2} = -25\cdot 20$, whence $u_{-2} = -563$.

Similarly we find $\cdot 0448\, u_{-3} = -69\cdot 22$, $u_{-3} = -1545$.

Forming the differences of the four values thus obtained we find as we expect that the second differences are nearly constant. We then build the table back in the manner described above and exemplified below, and by attaching three zero third differences inwards obtain u'_1, u'_2, u'_3.

In precisely the same manner, working from the opposite end of the data, we find in succession $u_{21} = 8208$, $u_{22} = 7402$, $u_{23} = 6008$, $u_{24} = 4026$, and form and build back a second table. Attaching three zero third differences inwards we obtain u'_{20}, u'_{19}, u'_{18}.

The final process of graduating the extended set by the k-series was arranged as shown on the accompanying page. A column was assigned for data and a column for each k_r. The products $k_r u_s$ (rapidly calculated by setting up the particular k_r on the multiplicand dials of an arithmometer and taking the u_s's as multipliers) were entered under the appropriate k_r and opposite the corresponding u_s. The graduated value u'_x was then obtained by summing the slanting lines of numbers which converge in the k_0 column opposite u_x. These are shown in red for u_{10} in our illustration.

Since six graduated values have already been found from the difference tables, there only remain fourteen to evaluate, and the necessary products lie in a trapezium, as shown.

In actual fact we found it slightly more accurate in the present example not to calculate the three graduated values at each end from the difference tables, but to find them by the series in the same way as all the rest. We did this because we were not taking precautions to secure the accuracy of final digits, and because the building back from third differences which may not be exact would produce an increasing error.

The extrapolated values, together with the difference tables by which they were found, are as follows:-

u	Δ	Δ^2	Δ^3
-8870			
	1787		
-7083		-161	
	1626		0
-5457		-161	
	1465		0
-3992		-161	
	1304		0
-2688		-161	
	1143		0
-1545		-161	
	982		0
-563		-161	
	821		0
258		-161	
	660		
918			

u	Δ	Δ^2	Δ^3
8428			
	-220		
8208		-586	
	-806		-2
7402		-588	
	-1394		0
6008		-588	
	-1982		0
4026		-588	
	-2570		0
1456		-588	
	-3158		0
-1702		-588	
	-3746		0
-5448		-588	
	-4334		0
-9782		-588	

[NOTE. It will be observed that the auxiliary values attached beyond the ends soon become negative and increasing. This fact is not at all significant. We must bear in mind that the added data are purely <u>auxiliary</u>, and that to try to give a meaning to them as rates of mortality or anything else is as futile as to try to give a similar meaning to the value of a Fourier series <u>outside</u> the range within which it represents any particular function.]

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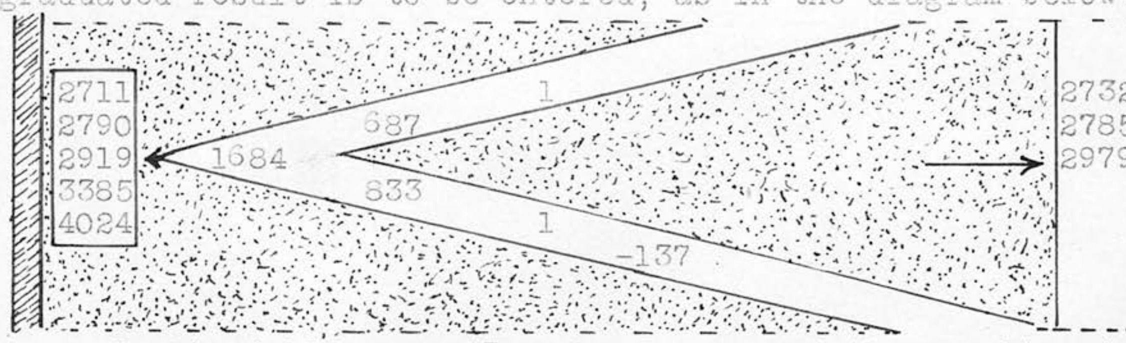
r	u_r	5770	2461	0003	-0343	-0072	0046	0027	0000	-0005	-0001	0001	u'_r	As by §10.
-8	-8870										1	-1		
-7	-7083									4	1	-1		
-6	-5457								0	3	1	-1		
-5	-3992							-11	0	2	0	0		
-4	-2688						-12	-7	0	1	0	0		
-3	-1545					11	-7	-4	0	1	0	0		
-2	- 563				19	4	-3	-2	0	0	0	0		
-1	258			0	-9	-2	1	1	0	0	0	0		
0	918		226	0	-31	-7	4	2	0	0	0	0		
1	1417	818	349	0	-49	-10	7	4	0	-1	0	0	1419	1416
2	1721	993	424	1	-59	-12	8	5	0	-1	0	0	1754	1753
3	2017	1164	496	1	-69	-15	9	5	0	-1	0	0	1928	1927
4	1848	1066	455	1	-63	-13	9	5	0	-1	0	0	1929	1931
5	2098	1211	516	1	-72	-15	10	6	0	-1	0	0	2061	2058
6	2226	1284	548	1	-76	-16	10	6	0	-1	0	0	2307	2305
7	2718	1568	669	1	-93	-20	13	7	0	-1	0	0	2622	2619
8	2711	1564	667	1	-93	-20	12	7	0	-1	0	0	2733	2732
9	2790	1610	687	1	-96	-20	13	8	0	-1	0	0	2786	2785
10	2919	1684	718	1	-100	-21	13	8	0	-1	0	0	2979	2979
11	3385	1953	833	1	-116	-24	16	9	0	-2	0	0	3407	3407
12	4024	2322	990	1	-138	-29	19	11	0	-2	0	0	3850	3848
13	4007	2312	986	1	-137	-29	18	11	0	-2	0	0	4128	4129
14	4524	2610	1113	1	-155	-33	21	12	0	-2	0	0	4698	4698
15	5746	3315	1414	2	-197	-41	26	16	0	-3	-1	1	5456	5454
16	5752	3319	1416	2	-197	-41	26	16	0	-3	-1	1	5744	5740
17	5828	3363	1434	2	-200	-42	27	16	0	-3	-1	1	6108	6105
18	7231	4172	1780	2	-248	-52	33	20	0	-4	-1	1	7119	7115
19	8146	4700	2005	2	-279	-59	37	22	0	-4	-1	1	8066	8062
20	8376	4833	2061	3	-287	-60	39	23	0	-4	-1	1	8429	8428
21	8208		2020	2	-282	-59	38	22	0	-4	-1	1		
22	7402			2	-254	-53	34	20	0	-4	-1	1		
23	6008				-206	-43	28	16	0	-3	-1	1		
24	4026					-29	19	11	0	-2	0	0		
25	1456						7	4	0	-1	0	0		
26	-1702							-5	0	1	0	0		
27	-5448								0	3	1	-1		
28	-9782									5	1	-1		
29	-14704										1	-1		

[The ordinary eye is apt to be confused and to lose the place in summing numbers in converging diagonal lines such as are illustrated on p. 43. This confusion can be entirely removed by using computing paper ruled in faint squares, entering the numbers in columns and rows of strictly equal width, and covering the sheet with a flat piece of stiff card or Bristol board having a V-shaped slot cut in it in such a way that only the numbers involved in the particular addition appear, the rest being blotted out from view by the card. For successive additions the card can be slid into place parallel to itself, its edge resting against a T-square or some similar raised straight edge. An arrowhead can be used to show where the graduated result is to be entered, as in the diagram below.



]

In the table on p.43 we have compared the results given by the new modified method with those previously obtained for the same graduation problem by the use of graduation coefficients, as performed in §10. We may take the latter to be accurate, since we have seen that they satisfy both the difference equation and the moment tests. There is a substantial agreement between the two sets of results, particularly when it is borne in mind that we took little care in the later method to safeguard the accuracy of the final digit. In no case is the discrepancy more than four units in the last place. For speed and simplicity of computation the new method is evidently superior; it requires vastly less apparatus than the old. The confirmation of our conjectures by the foregoing arithmetical test is so encouraging that we must now go back and investigate more closely what the coefficients of our graduating series really are.

§13. THE REAL GRADUATION FUNCTION.

The time has come for us to make a halt and consider whether the effort devoted hitherto to the graduation function G_x has not after all been concentrated on the wrong object. Attachment to that object has led us by a circuitous and rather wearisome route back towards our initial starting point, namely, the symbolic expansion of the operator

$$-\epsilon[(1-E^{-1})^6 - \epsilon E^{-3}]^{-1},$$

which we shall now examine in the light of later experience.

What is the vital objection to G_x? That it increases as x increases, in what tends to become a geometrical progression. There is a good reason for this. It is a well known algebraical fact that if we have a polynomial $F(x)$, and if $[F(x)]^{-1}$ is expanded in ascending powers of x^{-1}, then the ratio of the $(r+1)^{th}$ ~~######~~ coefficient to the r^{th} tends in general to a definite limit: this limit is the numerically greatest root of the equation $F(x) = 0$. Now inspection of the polynomial $(1-z)^6 - \epsilon z^3$ shows that as long as ϵ is positive, not zero, there is at least one root numerically greater than zero. This is why the G_x's increase with x. [E.g., if $\epsilon = 10$, the largest root of $(1-z)^6 - \epsilon z^3 = 0$ is found to be 3·89789, while from our table of G_x for $\epsilon = 10$ on p.21 we have

$G_{22} \div G_{21} = 3{\cdot}89836$.] The question arises, can we expand $[(1-z)^6 - \epsilon z^3]^{-1}$ in a series in which the successive coefficients will decrease?

Now that the graduation coefficients have shown the way, we see clearly the answer to this question, and with it the crux of the whole matter. The series of which the graduation function G_x is a typical coefficient, whether considered as a series in powers of z or of z^{-1}, is a Taylor series, which is divergent for $z = 1$, the very value in which we are interested, because of singularities within and without the circle of radius unity in the plane of the complex variable. What is suggested by the symmetrically disposed sequence of limiting graduation coefficients is that we should expand the reciprocal of the polynomial corresponding to our operator in a Laurent series, convergent in the annulus which includes the circumference of the circle with centre the origin and of unit radius. It is easy to show that in such an expansion the coefficients of like positive and negative powers will be the same.

Since $[(1-z)^6 - \epsilon z^3] = 0$ is a reciprocal equation, to each root α there corresponds a root α^{-1}. Obviously, unless $\epsilon = 0$, there are no roots of modulus unity. Hence so long as ϵ is positive we have three roots of the equation, α, β, γ, say, comprised within the circle in the Argand plane of centre the origin and unit radius, and the remaining three $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$, outside the same circle. If we take $|\alpha| > |\beta| > |\gamma|$, then it follows that $|\alpha^{-1}| < |\beta^{-1}| < |\gamma^{-1}|$, and $[(1-z)^6 - \epsilon z^3]^{-1}$ can be expanded as a series $\sum_{-\infty}^{\infty} k_r z^r$, convergent within the annulus $|\gamma^{-1}| > |z| > |\alpha|$.

By Laurent's Theorem the coefficients k_r are contour integrals, and from the reciprocal nature of $(1-z)^6 - \epsilon z^3$ we readily have

$$k_r = k_{-r} = \frac{1}{2\pi i}\int_C t^{r-1}\left[(1-t)^6 - \epsilon t^3\right]^{-1} dt,$$

where the contour C may be taken as the circle of unit radius and centre the origin. The three poles within being simple poles, this gives

$$k_r = k_{-r} = \left[\frac{\alpha^{r-1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\alpha^{-1})(\alpha-\beta^{-1})(\alpha-\gamma^{-1})} + \text{two similar}\right.$$

fractions obtained by interchanging β and γ in turn with α.]

The form of this result shows us that another way of arriving at the same expansion would have been to split up the function to be expanded into partial fractions. Three of those partial fractions could be expanded in convergent series of positive powers of z and the other three in convergent series of negative powers of z. Because of the reciprocal properties of the roots the coefficients of the powers in the first three series would be the same as those in the second three, and addition of the six series would give the desired expansion.

On p. 48 are shown the locations of the roots of the equation

$$\left[(1-z)^6 - \epsilon z^3\right] = 0$$

for the case $\epsilon = 1$.

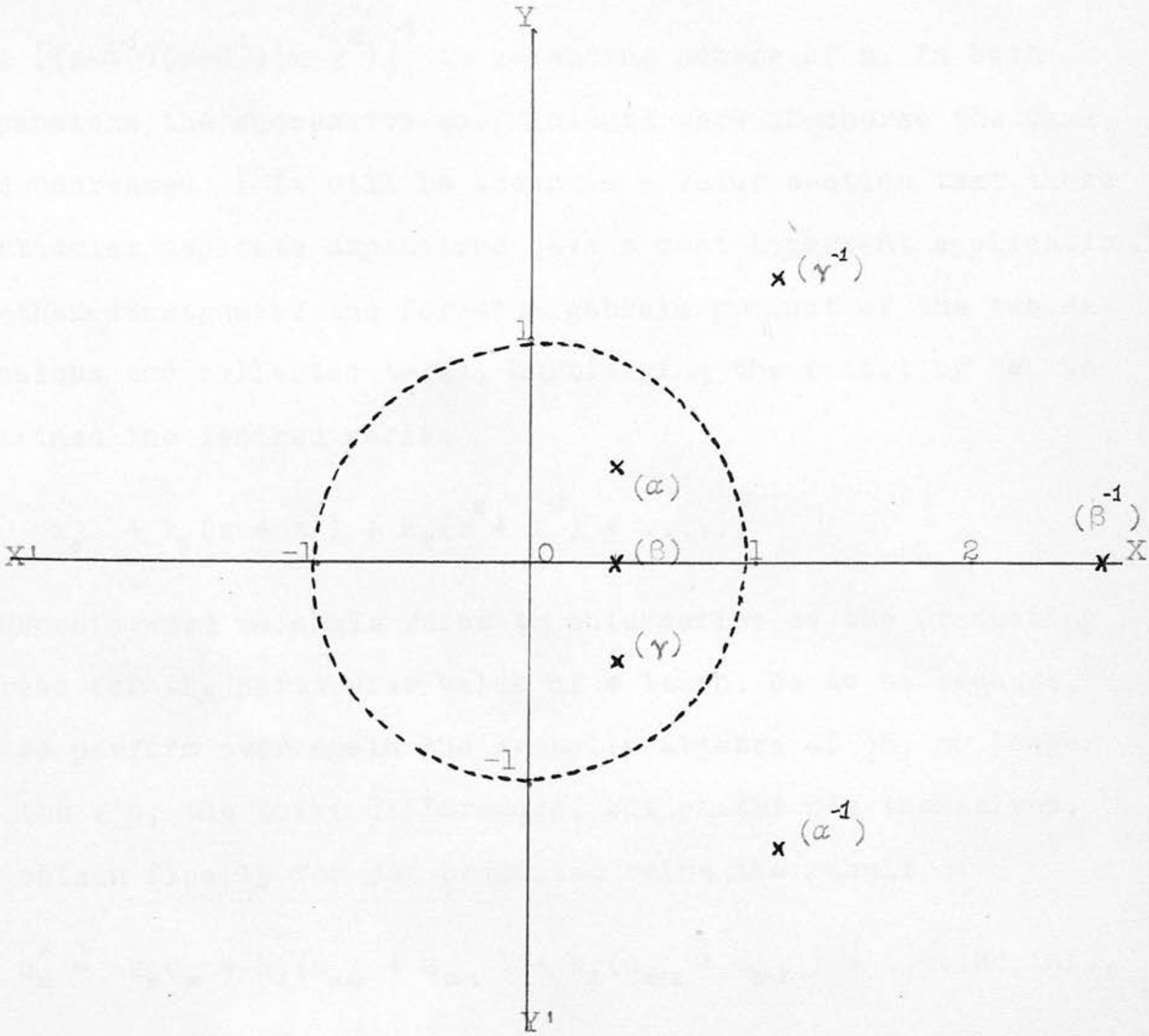


Diagram showing the positions of the six roots of the equation

$$(1-z)^6 - \epsilon z^3 = 0.$$

They are: $\alpha = \cdot 38 + \cdot 445i$, $\beta = \cdot 38$, $\gamma = \cdot 38 - \cdot 445i$;

$\alpha^{-1} = 1\cdot 12 - 1\cdot 31i$, $\beta^{-1} = 2\cdot 62$, $\gamma^{-1} = 1\cdot 12 + 1\cdot 31i$.

In the actual calculation of k_r, since an arithmometer was available, we did not evaluate by the formula given above, but proceeded as follows:-

We expanded $[(z-\alpha)(z-\beta)(z-\gamma)]^{-1}$ in ascending powers of z^{-1}, and $[(z-\alpha^{-1})(z-\beta^{-1})(z-\gamma^{-1})]^{-1}$ in ascending powers of z. In both expansions the successive coefficients were of course the same, and decreased. [It will be shown in a later section that these particular separate expansions have a most important application.] We then constructed the formal algebraic product of the two expansions and collected terms. Multiplying the result by $-\epsilon$, we obtained the desired series

$$k_0 + k_1(z + z^{-1}) + k_2(z^2 + z^{-2}) + \ldots\ldots$$

Henceforward we shall refer to this series as the Graduating Series for the particular value of ϵ taken. We do so because, if we perform over again the symbolic algebra of §5, no longer on the v's, the third differences, but on the u's themselves, we obtain finally for our graduated value the result

$$u'_x = k_0 u_x + k_1(u_{x+1} + u_{x-1}) + k_2(u_{x+2} + u_{x-2}) + \ldots\ldots \text{ad inf.},$$

where it is supposed that the u's are extended beyond the ends of the set given, by repeated appeal to the fundamental difference equation.

Clearly k_x , which has arisen in the same manner as G_x, is destined from now onwards to play the part previously assigned to G_x. It is the real Graduation Function.

It will have been observed that in naming the coefficients k_r we seem to have taken for granted that they are the same as the k_r's of §§11,12, the limits of $g_{r,s}$ as $n \to \infty$. As a matter of fact they are the same, but the proof requires some tedious algebra, which we shall omit, since G_x has now become of only secondary interest. We shall merely show how to calculate k_r for a given value of ϵ by an appropriate factorization of $(1-z)^6 - \epsilon z^3$, illustrating the method for $\epsilon = 10$ and comparing the results with those in the tenth row of the table of graduation coefficients for $n = 20$, $\epsilon = 10$, §9, p. 29.

§14. MODE OF CALCULATION OF k_x.

First we factorize $(1-z)^6 - \epsilon z^3$ into two cubic factors, each of which has roots the reciprocals of those of the other. To do this we assume the factors to be $z^3 - az^2 + bz - c$, $\frac{1}{c}(cz^3 - bz^2 + az - 1)$. Equating coefficients, we have three equations in a,b,c, of which the first two are

$$b = c(6 - a), \quad \ldots\ldots\ldots\ldots\ldots \quad (i)$$

$$bc + ab + a = 15c, \text{ which by (i) yields}$$

$$a^2 + a(c - 6 - \tfrac{1}{c}) + (15 - 6c) = 0. \qquad (ii)$$

Instead of using the third of the three equations, compute c accurately by the following formula:-

$$c = (1 + \tfrac{1}{2}\epsilon^{\frac{1}{3}} - \sqrt{q})\left\{1 - \tfrac{1}{2}\epsilon^{\frac{1}{3}} + \tfrac{1}{4}\epsilon^{\frac{2}{3}} + \sqrt{\tfrac{p}{q}} - (1 - \tfrac{1}{4}\epsilon^{\frac{1}{3}})\sqrt{[2\sqrt{\tfrac{p}{q}} - q]} - \tfrac{\sqrt{3}}{4}\epsilon^{\frac{1}{3}}\sqrt{[2\sqrt{\tfrac{p}{q}} + q]}\right\},$$

where $p = \epsilon + \frac{1}{64}\epsilon^2$, $q = \epsilon^{\frac{1}{3}} + \frac{1}{4}\epsilon^{\frac{2}{3}}$.

We now substitute the value of c in (ii) and solve for a, taking the lesser of the two roots. Then substituting the value of a in (i) we obtain b.

Now we expand $(z^3 - az^2 + bz - c)^{-1}$ in ascending powers of z^{-1}. The easiest way is ordinary long division, using detached coefficients. Suppose we obtain $z^{-3} + m_1 z^{-4} + m_2 z^{-5} + \ldots$ Forming the product $(z^3+m_1z^4+m_2z^5+\ldots)(z^{-3}+m_1z^{-4}+m_2z^{-5}+\ldots)$ and collecting terms, we have, on multiplying by ϵc,

$$k_0 = \epsilon c(1+m_1^2+m_2^2+\ldots) = \epsilon c\sum_{s=0}^{\infty} m_s^2, \text{ where } m_0 = 1.$$

$$k_1 = \epsilon c\sum_{s=0}^{\infty} m_s m_{s+1},$$

$$k_2 = \epsilon c\sum_{s=0}^{\infty} m_s m_{s+2}, \text{ and generally}$$

$$k_r = \epsilon c\sum_{s=0}^{\infty} m_s m_{s+r}.$$

To illustrate this we shall take $\epsilon = 10$.

The formula of the previous page gives $c = \cdot 044796106..$, and we find from the other equations $a = \cdot 53089940..$, $b = \cdot 24499441...$

Now using $1, -a, b, -c$ as detached coefficients and dividing into unity we obtain $m_1, m_2, m_3..$ in succession as

$\cdot 53089940$, $\cdot 03685976$, $-\cdot 06570246$, $-\cdot 02012961$, $\cdot 00706111$, $\cdot 00573717$, $\cdot 00041420$, $-\cdot 00086936$, $-\cdot 00030602$, $\cdot 00006907$, $\cdot 00007270$, $\cdot 00000797$, $-\cdot 00001049$, $-\cdot 00000426$, $\cdot 00000067$, $\cdot 00000092$, $\cdot 00000014....$

Substituting these values in the formulae for the k's given above and retaining six places of decimals we have finally the values of k_r tabulated on the next page, beside which we have put the values of $g_{r,s}$ from the tenth column of the table of graduation coefficients, for the sake of comparison. It will be seen that the agreement is remarkable, and that the earlier conjectures have been confirmed in a gratifying manner.

r	k_r	$g_{r,s}$ in §9.
0	·576982	·5770
1	·246051	·2461
2	·000293	·0003
3	-·034279	-·0343
4	-·007248	-·0072
5	·004563	·0046
6	·002663	·0027
7	-·000029	·0000
8	-·000463	-·0005
9	-·000120	-·0001
10	·000049	·0001

r	k_r
11	·000034
12	·000001
13	-·000006
14	-·000002
15	·000001
16	·000000

Corresponding to the moment tests by which we verified the accuracy of $g_{r,s}$ there are similar tests for k_r . If we graduate the infinite sequence .....,1,1,1,1,..... we have the test that the infinite series

$$M_0 = k_0 + 2(k_1 + k_2 + k_3 + \ldots\ldots)$$

should converge to the sum 1. The sum of this series as far as the seventeen terms given in the above table is ·999998.

If we graduate the sequence .....,-2,-1,0,1,2,3,.... no new fact emerges, but if we graduate the sequence $\ldots 2^2, 1^2, 0, 1^2, \ldots$ we deduce the more stringent test that

$$M_2 = k_0 n^2 + 2[k_1(n^2+1) + k_2(n^2+2^2) + k_3(n^2+3^2) + \ldots]$$

should converge to the sum n^2. If we put n = 1 and test the values found in the table above as far as the seventeen entries permit, we obtain the sum ·999780. The forcing of final digits easily accounts for the small discrepancy visible here.

§15. THE MODE OF CALCULATION OF i_x.

We observe at this stage that the method of graduation put forward in Professor Whittaker's second paper could now be used almost unchanged, with the substitution throughout of k_x for G_x. We should find, however, that we had to determine the three added third differences at each end by means of six, not three, linear equations. The reason that there were only three equations before was that the three initial G_x's, G_0, G_1, G_2, vanished, while none of the k_x's vanish.

There is in any case a practical objection to the graduation of third differences. When we have found these third differences, we have to build back the table by summing to get the graduated rates themselves. This is theoretically sound, but actually we should find that small deviations due to the forcing of last digits would produce considerable errors in the middle of the set of results. Though these errors could be diminished by using more significant digits throughout the work, it seems easier to employ a series by which we can extrapolate u's at the ends.

The coefficients i_r in the extrapolating series are also the coefficients of a symbolic expansion, almost as simple as the earlier one from which the k's were derived. We shall find

$$1 - \frac{c(z-1)^3}{z^3 - az^2 + bz - c} = i_0 + i_1 z^{-1} + i_2 z^{-2} + \ldots\ldots :$$

this theorem we conjectured from an inspection of the entries in the first row of the table of $g_{r,s}$ for $n = 20$, $\epsilon = 10$, but it was some time before we found the concise proof which the simplicity of the result demands. It may be proved thus:-

The condition under which the theorem is to hold is that $v_{x-4}, v_{x-5}, v_{x-6}, \ldots$ are to be zero.

Now we showed in §5 that

$$u'_x = \left[\frac{-\epsilon E^{-3}}{(1-E^{-1})^6 - \epsilon E^{-3}}\right]u_x \text{ . A different form of this is}$$

$$u'_x - u_x = -\left[\frac{(1-E^{-1})^6}{(1-E^{-1})^6 - \epsilon E^{-3}}\right]u_x$$

$$= \left[\frac{c}{E^{-3}-aE^{-2}+bE^{-1}-c}\right]\left[\frac{(E-1)^3}{E^3-aE^2+bE-c}\right]v_{x-3},$$

since $(1-E^{-1})^3 u_x = v_{x-3}$.

If we expand the second operating factor in powers of E^{-1} and operate on v_{x-3} we obtain v_{x-3} plus terms in $v_{x-4}, v_{x-5}, \ldots$. Since the latter by hypothesis contribute nothing, we have then

$$u'_x - u_x = \left[\frac{c}{E^{-3}-aE^{-2}+bE-c}\right]v_{x-3},$$

$$\text{i.e.} \quad u'_x = \left[1 - \frac{c(E^{-1}-1)^3}{E^{-3}-aE^2+bE-c}\right]u_x,$$

which proves the theorem.

Taking $\epsilon = 10$, we have the values of i_r given in the table on the following page. They are seen to be in complete agreement with the first row of graduation coefficients in the table for $n = 20$, $\epsilon = 10$, §9, p. 29.

[NOTE. The theorem was conjectured in the first place from observing that ·9552, the first entry in the row, = 1 - ·0448, i.e. 1-c, and that ·1106 ÷ ·0448 = 2·4691 approx., i.e. 3-a.]

r	i_r	$g_{r,s}$ in §9.
0	·955204	·9552
1	·110606	·1106
2	–·064693	–·0647
3	–·018654	–·0187
4	·010901	·0109
5	·007459	·0075
6	·000454	·0005
7	–·001098	–·0011
8	–·000360	–·0004
9	·000098	·0001
10	·000091	·0001
11	·000008	·0000
12	–·000014	
13	–·000005	
14	·000001	
15	·000001	

As before we have moment tests. The accuracy of our work is confirmed here by the values of the three infinite series

$M_0 = \cdot 999999$, $M_1 = \cdot 999990$, $M_2 = \cdot 999888$.

[It is evident that the limits of the successive graduation coefficients, not merely of the first row or of the middle row, but of <u>any</u> row of a matrix, such as the r^{th}, as $n \longrightarrow \infty$, are coefficients in symbolic expansions similar to those we have just given, but no doubt of more complicated appearance. We did not pause to determine their exact form, since we have found the only two that are of practical importance.]

§16. THE CHOICE OF THE PARAMETER ϵ.

We now have complete apparatus for the graduation of a large number of data, provided only that we can find a method for determining the most suitable value of ϵ for a given set of ungraduated data, and more generally for determining what values of ϵ are likely to be required in practice. It seems that only actual trial and comparison of numerous results will settle the second question, but our own experience led to the conclusion, based upon the inspection of the effect of graduation on sets of data derived from statistics of mortality, that the values of ϵ likely to be of service would lie between $\epsilon = \cdot01$ and $\epsilon = \cdot25$, and that of these values $\epsilon = \cdot05$ and $\epsilon = \cdot1$ would be of standard application. The only reasonable way of settling the first question, that of which value of ϵ to use for a particular set of data, seemed to be to observe the reduction brought about in the third differences by a standard value of ϵ, such as $\epsilon = \cdot1$, and to judge from the result what value of ϵ to take. In fact we form a table of third differences of the ungraduated data, select one or two of the largest and most irregular third differences appearing somewhere near the middle of the set, at any rate sufficiently far from the ends to allow of the convergence of the k-series having full play, then by means of the k-series find the corresponding graduated third differences, and from them decide whether the reduction is sufficient or insufficient. When we have calculated the coefficients of the graduating series for five selected values of ϵ, $\cdot01, \cdot05, \cdot1, \cdot25$ and 1, we shall show how this method of fixing upon ϵ works when applied to actual tables of rates of mortality.

§17. TABLES OF THE TWO SERIES USED IN GRADUATION.

In the present section are shown the details of the calculation of the extrapolating and graduating series, carried out in the manner described in §14. We shall change the i-series a little, in order to avoid division by c. Instead of writing

$u_0' = u_0 = i_0 u_0 + i_1 u_1 + i_2 u_2 + \ldots\ldots$, we shall write

$u_0 = j_1 u_1 + j_2 u_2 + j_3 u_3 + \ldots$, where it is easily seen that

$$1 - \frac{(z-1)^3}{z^3 - az^2 + bz - c} = j_1 z^{-1} + j_2 z^{-2} + j_3 z^{-3} + \ldots\ldots, \text{ i.e.}$$

that the j's can be found directly by taking the third differences of the m's.

The various constants entering into the calculations for each ϵ are shown by the following table:

ϵ	·01	·05	·1	·25	1
$\epsilon^{\frac{1}{3}}$	·21544347	·36840315	·46415888	·62996052	1·0
$\epsilon^{\frac{2}{3}}$	·04641589	·13572088	·21544347	·39685026	1·0
p	·01000156	·05003906	·10015625	·25097656	1·015625
q	·22704744	·40233337	·51801970	·72917309	1·25
$\sqrt{q}$	·47649495	·63429754	·71973585	·85391633	1·11803399
$\sqrt{\frac{p}{q}}$	·20988221	·35266435	·43970954	·58667998	·90138782
$\sqrt{(2\sqrt{\frac{p}{q}}-q)}$	·43899544	·55045011	·60116502	·66647345	·74348883
$\sqrt{(2\sqrt{\frac{p}{q}}+q)}$	·80424614	1·05245525	1·18213315	1·37932340	1·74721940
a	2·08450963	1·81525235	1·64709917	1·47961022	1·13847718
b	1·54073682	1·23056884	1·09027170	·90276369	·62616456
c	·39349779	·29406047	·25232509	·19970926	·12880009

The division , $1 \div (z^3 - az^2 + bz - c)$, carried out to a large number of terms in the quotient for each ϵ, gives $z^{-3} + m_1 z^{-4} + m_2 z^{-5} + \ldots\ldots$, the values of m_r being :

r	m_r	m_r	m_r	m_r	m_r
0	1·0	1·0	1·0	1·0	1·0

r	m_r	m_r	m_r	m_r	m_r
1	2·08450963	1·81525235	1·67909917	1·47961022	1·13847718
2	2·80444358	2·06457225	1·72910232	1·28638271	·66996573
3	3·02770670	1·80798712	1·32498505	·76746385	·17866672
4	2·81062422	1·27514855	·76326882	·26964934	-·06946486
5	2·29741641	·69697288	·27330624	-·03694007	-·10466728
6	1·64995031	·22768113	-.03893514	-·14481690	-·05265259
7	1·05597768	-·06940362	-·17076215	-·12707295	-·00335191
8	·45806714	-·20121021	-·17531468	-·06466027	·01567196
9	·05473579	-·21288936	-·11801790	-·00987643	·01315936
10	-·19596317	-·15925376	-·05011070	·01838203	·00473667
11	-·31257215	-·08627872	·00029446	·02320107	-·00082878
12	-·32809358	-·02324729	·02534983	·01576150	-·00221455
13	-·27943388	·01714196	·02959965	·00604686	-·00139218
14	-·19997320	·03435321	·02213685	-·00064844	-·00030505
15	-·11541609	·03442928	·01129471	-·00327061	·00023921
16	-·04243649	·02526461	·00229849	-·00304623	·00028404
17	·01067754	·01359596	-·00286923	-·00168414	·00013430
18	·04222491	·00371455	-·00447376	-·00039501	·00000585
19	·05486828	-·00255859	-·00380368	·00032756	-·00004085
20	·05351758	-·00521747	-·00223313	·00050492	-·00003287
21	·04363574	-·00523020	-·00073145	·00037248	-·00001109
22	·03009316	-·00382605	·00024678	·00016073	·00000269
23	·01655734	-·00204338	·00064838	·00000240	·00000577
24	·00531886	-·00053903	·00063507	-·00006716	·00000346
25	-·00258170	·00041095	·00042171	-·00006944	·00000068
26	-·00706126	·00080841	·00017929	-·00004163	-·00000066
27	-·00864858	·00080326	.00000151	-·00001232	-·00000073

r	m_r	m_r	m_r	m_r	m_r
28	-·00816440	·00058416	-·00008652	·00000548	-·00000033
29	-·00647217	·00030965	-·00010169	·00001092	-·00000001
30	-·00431531	·00007945	-·00007604	·00000875	·00000011
31	-·00223607	-·00006505	-·00003864	·00000418	·00000015
32	-·00055913	-·00012479	-·00000764	·00000046	·00000010
33	·00058163	-·00012312	·00001011	-·00000134	·00000003
34	·00119399	-·00008906	·00001556	-·00000157	-·00000001
35	·00137272	-·00004686	·00001318	-·00000102	-·00000002
36	·00125070	-·00001167	·00000772	-·00000036	-·00000001
37	·00096193	·00001029	·00000252	·00000008	·00000000
38	·00061831	·00001926	-·00000086	·00000024	·00000001
39	·00029894	·00001887	-·00000224	·00000022	·00000000
40	·00004901	·00001358	-·00000218	·00000013	....
41	-·00011513	·00000709	-·00000144	·00000004	
42	-·00019787	·00000171	-·00000061	....	
43	-·00021578	-·00000164	·00000000		
44	-·00019023	-·00000300	....		
45	-·00014194	-·00000293			
46	-·00008770	-·00000211			
47	-·00003898	-·00000110			
48	-·00000198	....			
49	·00002142				
50	·00003236				
51	·00003367				
52	·00002876				
53	·00002080				
54	·00001230				

r	m_r
55	·00000491
56	−·00000054
57	−·00000386
58	−·00000529
59	−·00000529
60	−·00000440

By forming third differences of the m's, which is most easily done by machine with the coefficients 1,-3,3,-1, we have the j's, and by the formulae of §14 we find the k's. As the table for each value of ϵ was completed, it was subjected to two tests of accuracy, the test by moments and the test by central differences, for it is obvious that we must have $\delta^6 j_r = \epsilon j_r$ and $\delta^6 k_r = \epsilon k_r$. It may be asserted as a result of these tests that the values given in the tables, to six places of decimals, are in no case in error by more than a unit of the last place.

It is interesting to follow the changes that take place in j_r and k_r as r increases. Their mode of decrease is not uniform: it is oscillatory, as will be seen from the diagram on page 61, recalling in appearance the amplitude of a damped vibration. Both the length of the period of oscillation and the ratio of the decrease of amplitude in succeeding periods are characteristic of the particular value of ϵ involved; as ϵ increases the period becomes shorter and shorter until as ϵ tends to infinity the graph tends to become the horizontal axis and a unit length of the vertical axis measured positively from the origin. The periodicity of the coefficients is of course due to the fact

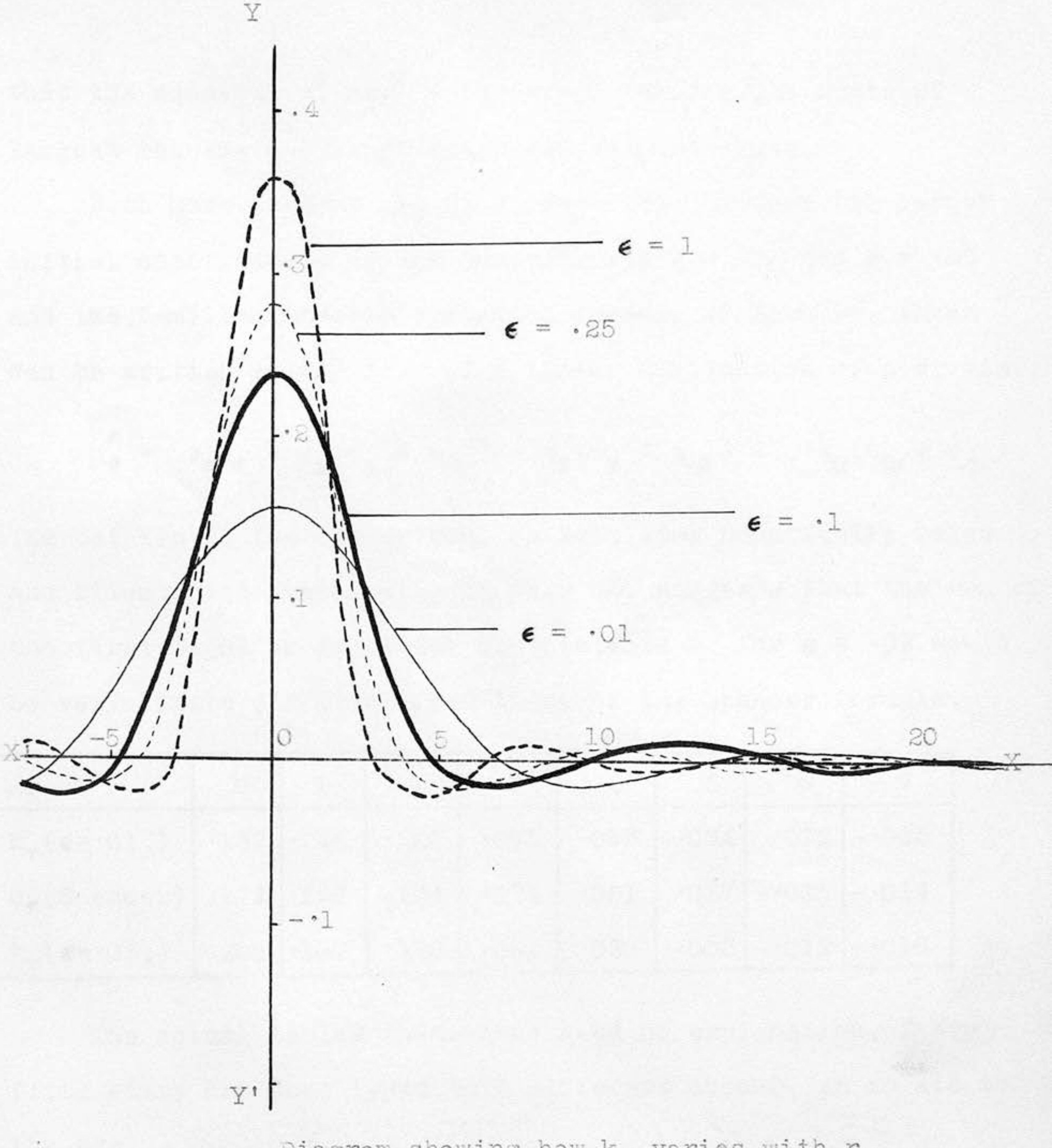


Diagram showing how k_r varies with r.

XOX' is the axis of r,

YOY' .. .. k.

that the equation $z^3 - az^2 + bz - c = 0$ has for its roots of largest modulus a pair of conjugate complex roots.

Even more interesting is a comparison between the larger initial coefficients of the k-series for $\epsilon = \cdot 01$ and $\epsilon = \cdot 05$ and the familiar 21-term summation formula of Spencer, which can be written in the form of a linear combination of u's, viz.,

$$u_0' = s_0 u_0 + s_1(u_1 + u_{-1}) + s_2(u_2 + u_{-2}) + \ldots + s_{10}(u_{10} + u_{-10}).$$

The details of the comparison, as tabulated numerically below and illustrated graphically on page 63, suggests that the run of the first eight or so of the coefficients k for $\epsilon = \cdot 02$ would be very little different from those of the Spencer formula.

r	0	1	2	3	4	5	6	7
$k_r(\epsilon = \cdot 01.)$	·157	·148	·125	·095	·063	·034	·012	−·003
s_r(Spencer)	·171	·163	·134	·094	·051	·017	−·006	−·014
$k_r(\epsilon = \cdot 05.)$	·208	·187	·140	·084	·036	·003	−·019	−·016

The actual tables themselves need no explanation. Every fifth entry has been typed in a different colour, as an aid to the eye.

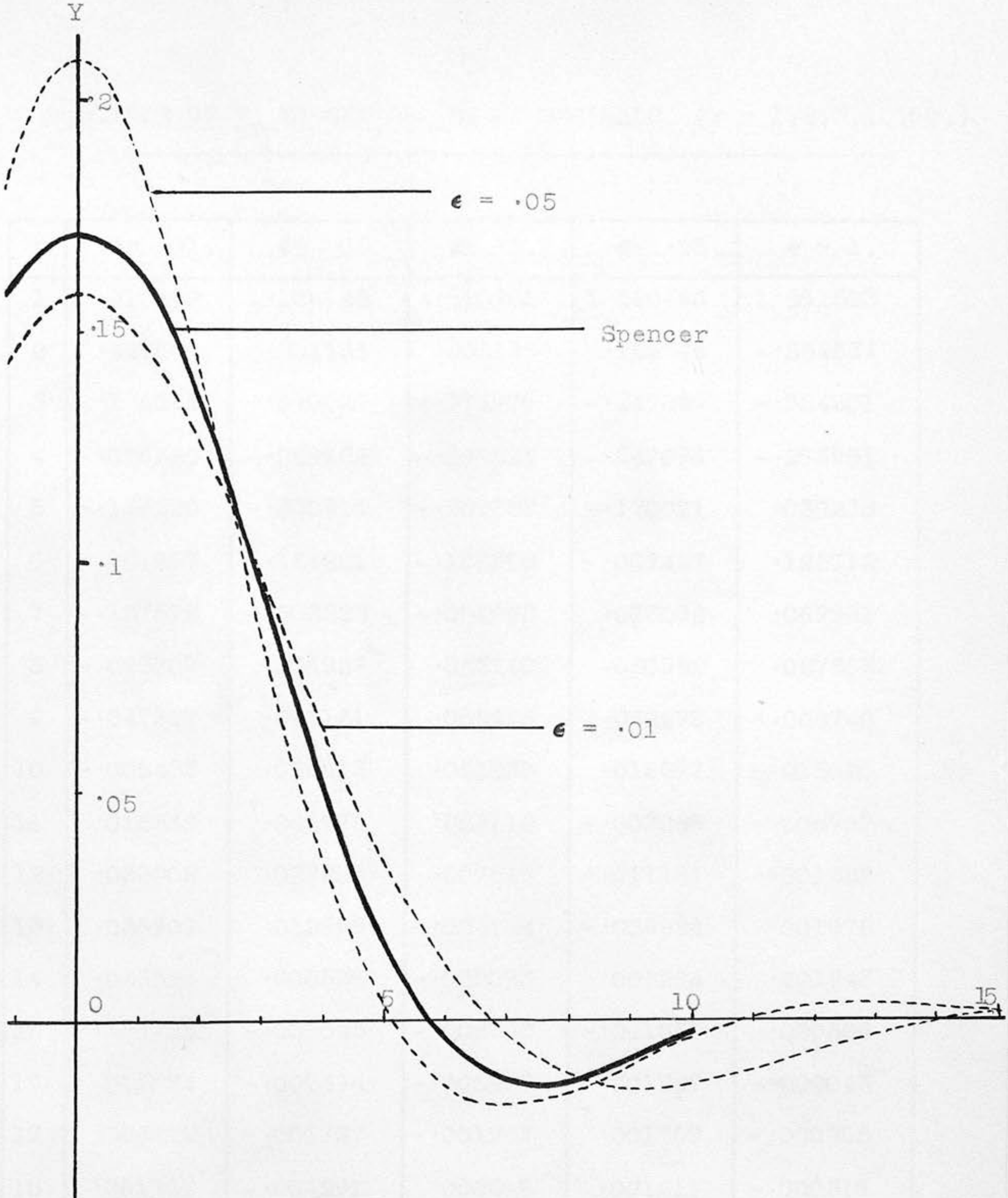


Diagram showing comparison between the graduating coefficients corresponding to the Spencer summation formula and the coefficients k_r of the present method for $\epsilon = \cdot 01$ and $\epsilon = \cdot 05$.

TABLE OF j_r TO SIX PLACES OF DECIMALS. (r = 1,2,3,...60.)

r	ϵ = ·01.	ϵ = ·05.	ϵ = ·1.	ϵ = ·25.	ϵ = 1.
1	·915490	1·184748	1·320902	1·520390	1·861523
2	·449085	·381185	·308195	·152348	-·254534
3	·132095	-·060027	-·174976	-·346846	-·584201
4	-·056325	-·229652	-·296521	-·347096	-·265951
5	-·144220	-·230916	-·229353	-·170021	·030238
6	-·161867	-·154221	-·105968	-·007487	·125712
7	-·137372	-·063323	-·002693	·073092	·089931
8	-·093709	·006929	·053140	·080952	·027563
9	-·047377	·045151	·065425	·052298	-·008740
10	-·008433	·054813	·051239	·018897	-·015626
11	·018542	·045975	·028112	-·003086	-·008767
12	·033002	·029283	·007848	-·011181	-·001322
13	·036906	·012699	-·004544	-·009984	·001972
14	·033380	·000536	-·009093	-·005294	·001943
15	·025705	-·006043	-·008333	-·001055	·000808
16	·016674	-·007894	-·005225	·001227	-·000043
17	·008288	-·006737	-·001983	·001709	-·000305
18	·001701	-·004291	·000265	·001211	-·000216
19	-·002663	-·001821	·001289	·000494	-·000060
20	-·004910	-·000006	·001374	-·000021	·000027
21	-·005463	·000968	·000969	-·000235	·000041
22	-·004870	·001229	·000455	-·000230	·000022
23	-·003668	·001038	·000053	-·000133	·000003
24	-·002291	·000657	-·000162	-·000035	-·000005

TABLE OF j_r (continued)

r	ϵ= ·01.	ϵ= ·05.	ϵ= ·1.	ϵ= ·25.	ϵ= 1.
25	-·001041	·000276	-·000215	·000021	-·000005
26	-·000083	-·000002	-·000171	·000037	-·000002
27	·000529	-·000150	-·000094	·000029	·000000
28	·000821	-·000189	-·000025	·000013	·000001
29	·000863	-·000159	·000017	·000001	·000001
30	·000743	-·000100	·000032	-·000005	·000000
31	·000542	-·000041	·000029	-·000005	....
32	·000325	·000001	·000018	-·000003	
33	·000134	·000023	·000007	-·000001	
34	-·000008	·000029	-·000001	·000000	
35	-·000095	·000024	-·000004	·000001	
36	-·000133	·000015	-·000005	·000001	
37	-·000134	·000006	-·000003	·000000	
38	-·000112	·000000	-·000002	....	
39	-·000079	-·000004	·000000		
40	-·000045	-·000004	·000001		
41	-·000016	-·000004	·000001		
42	·000004	-·000002	·000001		
43	·000017	-·000001	....		
44	·000021	·000000			
45	·000021	·000001			
46	·000017	·000001			
47	·000011	·000001			
48	·000006	·000000			
49	·000002	-·000001			

TABLE OF j_r (continued)

r	ϵ= ·01.	ϵ= ·05.
50	-·000001	·000000
51	-·000003	·......
52	-·000003	
53	-·000003	
54	-·000003	
55	-·000002	
56	-·000001	
57	·000000	
58	·000000	
59	·000000	
60	·000001	
61	·000000	
62	....	

TABLE OF k_r TO SIX PLACES OF DECIMALS. (r =1,2,3,...52.)

r	ϵ= ·01.	ϵ= ·05.	ϵ= ·1.	ϵ= ·25.	ϵ= 1.
0	·156964	·207571	·234715	·277094	·360115
1	·148200	·187345	·205586	·229703	·260393
2	·125401	·139738	·141170	·135594	·104501
3	·094827	·084158	·072119	·048596	·002304
4	·062775	·035901	·019056	-·004631	-·029272
5	·034096	·002699	-·011013	-·023644	-·021309
6	·011668	-·014532	-·021069	-·021100	-·005634
7	-·003492	-·019143	-·018563	-·010797	·003159
8	-·011875	-·016074	-·010796	-·001651	·004379
9	-·014756	-·009894	-·003508	·003091	·002282
10	-·013844	-·003810	·001393	·003908	·000263
11	-·010795	·000533	·003394	·002662	-·000566
12	-·006979	·002747	·003295	·001023	-·000526
13	-·003363	·003209	·002184	-·000102	-·000198
14	-·000505	·002602	·000931	-·000547	·000024
15	·001382	·001583	·000013	-·000512	·000085
16	·002336	·000614	-·000441	-·000284	·000056
17	·002541	-·000067	-·000521	-·000068	·000014
18	·002242	-·000413	-·000390	·000054	-·000008
19	·001677	-·000485	-·000198	·000085	-·000011
20	·001042	-·000394	-·000039	·000062	-·000006
21	·000470	-·000238	·000052	·000027	·000000
22	·000034	-·000091	·000080	·000001	·000002

TABLE OF k_r (continued).

r	ϵ= ·01.	ϵ= ·05.	ϵ= .1.	ϵ= ·25.	ϵ = 1.
23	-·000243	·000013	·000068	-·000011	·000001
24	-·000374	·000064	·000040	-·000012	·000000
25	-·000392	·000075	·000013	-·000007	....
26	-·000336	·000060	-·000004	-·000002	
27	-·000244	·000036	-·000012	·000001	
28	-·000146	·000014	-·000011	·000002	
29	-·000059	-·000002	-·000007	·000001	
30	·000005	-·000010	-·000003	·000001	
31	·000044	-·000012	·000000	·000000	
32	·000061	-·000009	·000002	....	
33	·000061	-·000005	·000002		
34	·000051	-·000002	·000001		
35	·000036	·000000	·000001		
36	·000020	·000002	·000000		
37	·000007	·000002	....		
38	-·000002	·000001			
39	-·000008	·000001			
40	-.000010	·000000			
41	-·000009	....			
42	-·000008				
43	-·000005				
44	-·000003				
45	-·000001				
46	·000001				

TABLE OF k_r (continued)

r	$\epsilon = \cdot 01$.
47	·000001
48	·000002
49	·000001
50	·000001
51	·000001
52	·000000
53	....

§18. APPLICATION OF THE TABLES TO ACTUAL GRADUATION.

The whole of the preceding work has been based upon the early assumption that the measure of precision is constant over the entire set of data to be graduated. This assumption is of course not supported by actual experience; for example, in tables of rates of mortality the number of observations from which the rate at each age is calculated, the "exposed to risk", as it is termed, passes over a wide range of values, rising from zero to a maximum and then decreasing to zero again. Hence by assuming ϵ constant in a graduation of rates of mortality we are really attaching too much weight to the more unreliable data at the ends of the set, and the effect will be to represent the high rates at extreme old age too faithfully. To give a complete and satisfactory solution in accordance with the present theory it would be necessary to take account of the variable "exposed to risk", in fact to solve, no longer a central difference equation with constant coefficients, but a central difference equation with a variable coefficient, ϵ_x , say. We have to admit a lack of success in finding a practical solution to this problem. The attempt to proceed on symbolic lines similar to those employed earlier in this paper is frustrated immediately by the fact that ϵ_x and the operator E are not commutative. We have therefore to fall back on either of two methods; (i) to discard a certain number of data near the ends as unreliable, graduate the remainder under the assumption of a constant ϵ, and adjust the ends later; (ii) to graduate the exposed to risk first, then adjust the actual deaths (or cases of sickness, etc.) proportional to the graduated exposed to risk, then graduate these adjusted deaths, when the graduated rate of mortality may be found as the quotient of graduated deaths divided by graduated exposed to risk. The first of these alternatives is the more rapid, but also the more arbitrary; the second involves two graduations and one proportional adjustment and is therefore more tedious to perform, but to our mind is the more satisfactory, provided we make the graduated exposed to risk as smooth as we can consistent with a reasonable fidelity to the original. It may even be possible to find a smooth curve, such as , for example,

$$E_x = \exp [ax^3 + bx^2 + cx + d] ,$$

which shall agree sufficiently well with the given exposed to risk.

The graduations performed in the earlier sections of this paper involved small numbers of data; the object was not to effect a practical graduation, but to obtain some numerical material for examination. We shall now graduate more extended sets, and shall take for our purpose two tables showing the rates of mortality of annuitants (ultimate) in the United Kingdom during the period 1900-1920, both from Elderton and Oakley's report cited previously on p.33. We shall first graduate the table for male annuitants (Appendix of the report, p.88) between the ages 41-95, since the rates for higher ages than 95 are evidently unreliable.

The complete table as it appears in the report is given on the following pages. We have added the third differences.

From "Report on the results of an investigation of the mortality experience of life annuitants during the period 1900-1920", by W.Palin Elderton and H.J.P.Oakley, in the J.I.A. LIV (1923), p.43—:Appendix, p.88, Table XX.

TABLE XX.

ANNUITY EXPERIENCE, 1900-1920.

Unadjusted Data.

MALES. ULTIMATE.

Age (l.b.d.)	Deaths.	Exposed to Risk.	Rate of Mortality.	
20-39	17	1109·5	.....	
40	...	205·5	.....	
41	2	251	·00797	Δ^3
42	4	297	·01347	1839
43	3	334	·00898	- 736
44	5	388	·01289	- 688
45	8	448·5	·01784	1607
46	9	531	·01695	-3861
47	16	608·5	·02629	4914
48	5	689·5	·00725	-2165
49	7	780·5	·00897	180
50	9	918·5	·00980	- 428
51	12	1040	·01154	1118
52	12	1211	·00991	-1638
53	22	1367	·01609	1190
54	21	1533	·01370	- 267
55	26	1775·5	·01464	26

TABLE XX (continued)

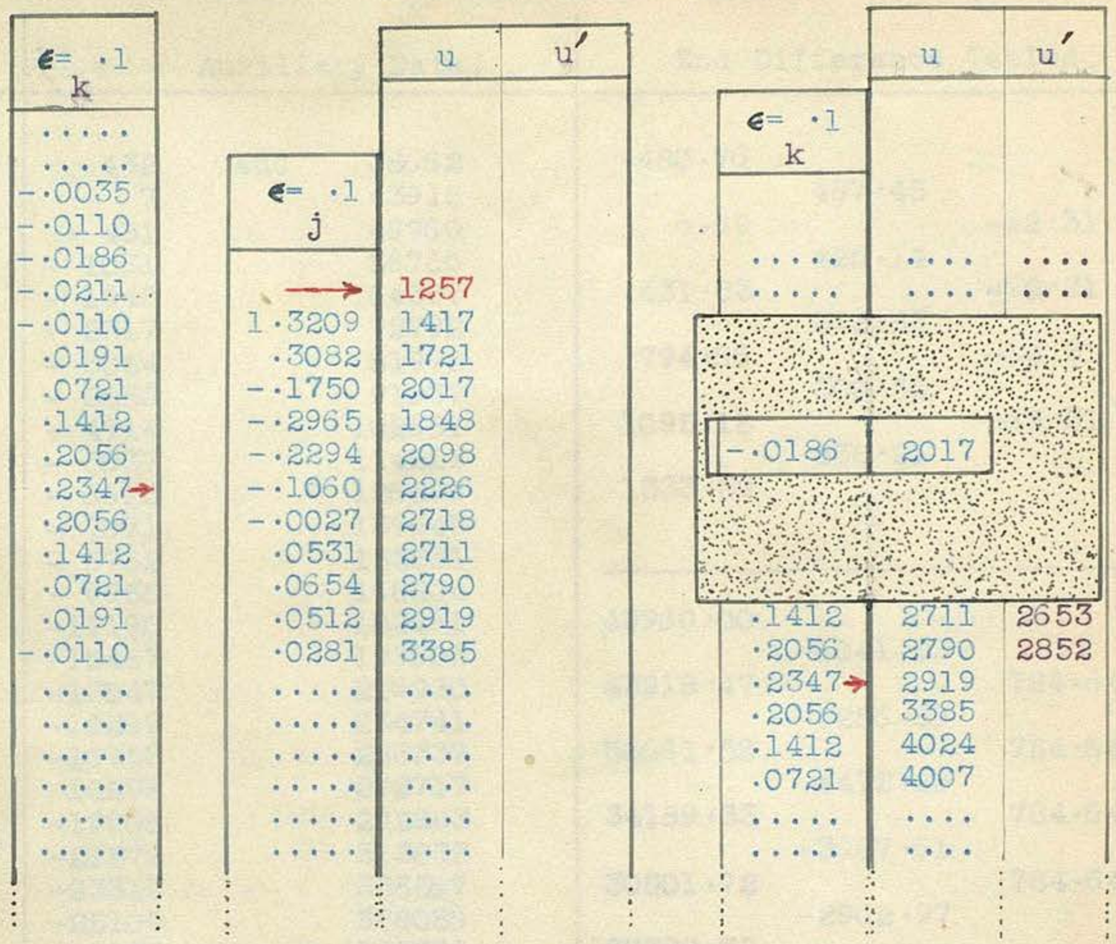
Age.	Deaths.	Exposed to Risk.	Rate of Mortality.	Δ^3
56	33	2032	·01624	359
57	43	2292	·01876	−1639
58	65	2520	·02579	1850
59	57	2721·5	·02094	117
60	68	2994	·02271	−1756
61	105	3253·5	·03227	227
62	110	3431	·03206	2906
63	87	3572·5	·02435	−3626
64	141	3691	·03820	2784
65	144	3855·5	·03735	−2598
66	195	3928·5	·04964	1206
67	199	4053·5	·04909	257
68	199	4167	·04776	− 644
69	205	4251	·04822	2897
70	193	4383·5	·04403	−3786
71	280	4364	·06416	395
72	307	4339	·07075	850
73	284	4192	·06775	1646
74	258	4053	·06366	365
75	295	3936·5	·07494	−5371
76	398	3782	·10524	4124
77	354	3510	·10085	− 414
78	342	3320	·10301	2427
79	325	3021	·10758	−7375
80	384	2766	·13883	7981

TABLE XX (continued)

Age.	Deaths.	Exposed to Risk.	Rate of Mortality.	Δ^3
81	301	2447	·12301	
				-3378
82	307	2194	·13993	
				74
83	299	1919	·15581	
				- 640
84	284	1657	·17139	
				701
85	254	1409	·18027	
				4176
86	223	1177	·18946	
				-12397
87	227	943	·24072	
				14283
88	150	714	·21008	
				- 9586
89	131	545	·24037	
				4874
90	95	403	·23573	
				- 6741
91	72	294	·24490	
				20244
92	43	214·5	·20047	
				-25201
93	50	164	·30488	
				13956
94	30	98	·30612	
				-10070
95	22	64	·34375	
				5963
96	13	41	·31707	
				5033
97	8	28	·28571	
				8450
98	6	20	·30000	
				-11903
99	4	9	·44444	
				-26668
100	3	5	·60000	
				85556
101	1	2	·50000	
102	1	1	1·00000	

A number of irregularities are apparent in the table, and these are brought into relief by the third differences. A trial graduation of two or three of the large and irregular third differences in the neighbourhood of age 80 shows that the value $\epsilon = \cdot 05$ reduces the large differences -5371 and 7981 to about 50; we therefore adopt this value of ϵ.

It is convenient here to describe the method by which the j-series and the k-series were employed in these graduations. For each ϵ we had two long narrow rectangular strips of Bristol board, one for the j's and one for the k's. On these the successive values of the respective j's or k's were written, vertically beneath one another, on lines a quarter of an inch apart, as in the illustration. The data to be graduated, or to have additional auxiliary data attached outside, were entered below each other on similar strips. Extrapolation and graduation were carried out by placing strips side by side, held by a paper-clip in alignment such that each u was opposite its appropriate j or k, and summing by machine the products of numbers on the same line. An arrowhead was used to indicate the place where the result had to be entered, and when this was done the extrapolating or graduating card was moved up one space. To avoid losing the place a piece of card with an oblong window cut in it was placed over the two adjacent strips in such a way as to leave visible only the two numbers whose product was required at the moment. With these devices, which could obviously be refined upon and improved ~~upon~~, all the processes of graduation were performed at a good uniform pace.



By means of a card of j-coefficients for ϵ = ·05 four auxiliary u's were extrapolated at each end, namely, 431·83, 6·69, -480·76 and -1030·51 at the lower ages, 38661·58, 43918·47 49960·00 and 56786·18 at the upper. As a check, the second differences ought to be constant; they prove to be nearly so, being -62·31, -62·30 and 784·64, 784·65 respectively. Accordingly we build the table outwards by repetition of the same second differences, obtaining the external auxiliary data shown below. Also we attach three more similar second differences inwards at each end and by building back obtain the six graduated values shown in red. Finally we enter the additional data on the ends of the strip and graduate the 49 remaining values by placing the card of k's alongside in the manner described above and forming and summing the products on a machine.

Auxiliary Data.

432	and	38662
- 7		43918
- 481		49960
- 1031		56786
- 1643		64397
- 2317		72792
- 3054		81975
- 3853		91937
- 4714		102687
- 5637		114221
- 6623		126539
- 7671		139643
- 8782		153531
- 9955		168204
-11190		183661
-12487		199903
-13847		216930
-15269		234741
-16753		253337
-18299		272717
-19908		292883
-21579		313833
-23313		335567
-25109		358086
-26967		381390
-28887		405479
-30870		430352
-32915		456010
......		......

End Difference Tables.

-480·76		
	487·45	
6·69		-62·31
	425·14	
431·83		-62·31
	362·83	
794·66		-62·31
	300·52	
1095·18		-62·31
	238·21	
1333·39		

49960·00		
	-6041·53	
43918·47		784·64
	-5256·89	
38661·58		784·64
	-4472·25	
34189·33		784·64
	-3687·61	
30501·72		784·64
	-2902·97	
27598·75		

The results of the graduation are given on the following pages, which show the ungraduated and graduated values, the ungraduated and graduated third differences, and the deviation of the graduated values from the ungraduated; also a second table showing the comparison of the expected with the actual deaths, with the deviations.

The exact total time occupied in graduating the 55 values was not taken, but it was certainly several hours. Of this time the greater part of an hour was taken up by attaching the end values and finding the first six graduated values.

TABLE XX GRADUATED.

Age.	u	u'	$\Delta^3 u$	$\Delta^3 u'$	u' - u
41	797	795	1839	-1	-2
42	1347	1095	- 736	-10	-252
43	898	1333	- 688	-21	+435
44	1289	1508	1607	2	+219
45	1784	1610	-3861	27	-174
46	1695	1618	4914	62	- 77
47	2629	1534	-2165	51	-1095
48	725	1385	180	24	+ 660
49	897	1233	- 428	0	+ 336
50	980	1129	1118	-12	+ 149
51	1154	1097	-1638	-21	- 57
52	991	1137	1190	-12	+ 146
53	1609	1237	- 267	- 7	- 372
54	1370	1376	26	- 8	+ 6
55	1464	1542	359	- 9	+ 78
56	1624	1728	-1639	- 3	+ 104
57	1876	1926	1850	12	+ 50
58	2579	2127	117	7	- 452
59	2094	2328	-1756	4	+ 234
60	2271	2541	227	10	+ 270
61	3227	2773	2906	3	- 454
62	3206	3028	-3626	-25	- 178
63	2435	3316	2784	-31	+ 881
64	3820	3640	-2598	-20	- 180
65	3735	3975	1206	13	+ 240
66	4964	4290	257	42	- 674
67	4909	4565	- 644	44	- 344
68	4776	4813	2897	23	+ 37

TABLE XX GRADUATED (continued)

Age.	u	u′	$\Delta^3 u$	$\Delta^3 u'$	u′ - u
69	4822	5076	-3786	- 8	+ 254
70	4403	5398	395	- 1	+ 995
71	6416	5802	850	19	- 814
72	7075	6280	1646	1	- 795
73	6775	6831	365	-37	+ 56
74	6366	7474	-5371	-54	+ 1108
75	7494	8210	4124	- 5	+ 716
76	10524	9002	- 414	32	- 1522
77	10085	9796	2427	38	- 289
78	10301	10587	-7375	38	+ 286
79	10758	11407	7981	49	+ 649
80	13883	12294	-3378	6	- 1589
81	12301	13286	74	-46	+ 985
82	13993	14432	- 640	-93	+ 439
83	15581	15738	701	-113	+ 157
84	17139	17158	4176	-119	+ 19
85	18027	18599	-12397	- 69	+ 572
86	18946	19948	14283	76	+ 1002
87	24072	21086	- 9586	173	- 2986
88	21008	21944	4874	269	+ 936
89	24037	22598	- 6741	286	- 1439
90	23573	23221	20244	223	- 352
91	24490	24082	- 25201	24	- 408
92	20047	25467	13956	13	+ 5420
93	30488	27599			- 2889
94	30612	30502			- 110
95	34375	34189			- 186
	488736	488686			

TABLE XX. COMPARISON OF ACTUAL AND EXPECTED DEATHS.

Age.	Exposed to Risk.	Actual Deaths.	Expected.	Deviation.
41	251	2	2	0
42	297	4	3	− 1
43	334	3	4	+ 1
44	388	5	6	+ 1
45	448·5	8	7	− 1
46	531	9	9	0
47	608·5	16	9	− 7
48	689·5	5	10	+ 5
49	780·5	7	10	+ 3
50	918·5	9	10	+ 1
51	1040	12	11	− 1
52	1211	12	14	+ 2
53	1367	22	17	− 5
54	1533	21	21	0
55	1775·5	26	27	+ 1
56	2032	33	35	+ 2
57	2292	43	44	+ 1
58	2520	65	54	− 11
59	2721·5	57	63	+ 6
60	2994	68	76	+ 8
61	3253·5	105	90	− 15
62	3431	110	104	− 6
63	3572·5	87	118	+ 31
64	3691	141	134	− 7
65	3855·5	144	153	+ 9
66	3928·5	195	169	− 26
67	4053·5	199	185	− 14

TABLE XX. COMPARISON OF ACTUAL AND EXPECTED DEATHS (continued)

Age.	Exposed to Risk.	Actual Deaths.	Expected .	Deviation.
68	4167	199	201	+ 2
69	4251	205	216	+ 11
70	4383·5	193	237	+ 44
71	4364	280	253	- 27
72	4339	307	272	- 35
73	4192	284	286	+ 2
74	4053	258	303	+ 45
75	3936·5	295	323	+ 28
76	3782	398	340	- 58
77	3510	354	344	- 10
78	3320	342	351	+ 9
79	3021	325	345	+ 20
80	2766	384	340	- 44
81	2447	301	325	+ 24
82	2194	307	317	+ 10
83	1919	299	302	+ 3
84	1657	284	284	0
85	1409	254	262	+ 8
86	1177	223	235	+ 12
87	943	227	199	- 28
88	714	150	157	+ 7
89	545	131	123	- 8
90	403	95	94	- 1
91	294	72	71	- 1
92	214·5	43	55	+ 12
93	164	50	45	- 5
94	98	30	30	0
95	64	22	22	0
		7720	7717	

Criticism of the results of this graduation will be reserved for the next section. We must mention, however, that before obtaining the results by the process just described we tried to obtain them by the original method of graduating third differences, which may seem an easier method because of the vanishing of ungraduated third differences beyond the termini with the exception of three at each end. On building the table back again, however, from each end, we found that the small deviations caused by forcing last digits had a profound effect in the middle of the table of rates of mortality; in fact there was a considerable discrepancy in the values obtained in the middle of the table, and the moment of zero order of the set as graduated was found to be in excess by a considerable amount. Instead of rectifying this by retaining more places of decimals throughout we found it more satisfactory to augment the rates themselves in the manner described and to graduate them directly.

We proceed to graduate a second table from the report of Elderton and Oakley, Table XV (Appendix to the report, p. 83). In view of the unreliable nature of the rates of mortality at advanced ages it was decided to graduate the table between the ages of 40 and 95. If the third differences in the appended facsimile of the table are examined, it will be seen that they are neither so large nor so irregular as those of the previous (Male Ultimate) graduation. This suggests that $\epsilon = \cdot 1$ may here be a suitable value to employ, and the trial graduation of the group of large and irregular third differences about age 72 confirms this. Accordingly we proceeded to graduate with this value exactly as in the previous example. The details are:—

TABLE XV. (from Elderton and Oakley's report: loc.cit., Appendix, p.83.)

ANNUITY EXPERIENCE, 1900-1920.

Unadjusted Data.

FEMALES. ULTIMATE.

Age.(l.b.d.)	Deaths.	Exposed to Risk.	Rate of Mortality.	Δ^3
40	3	546	·00549	
41	8	630·5	·01269	
				1408
42	5	717	·00697	
				934
43	2	830	·00241	
				-2214
44	8	958	·00835	
				1772
45	3	1131·5	·00265	
				- 297
46	4	1319·5	·00303	
				- 517
47	10	1533·5	·00652	
				55
48	14	1760	·00795	
				422
49	16	2033·5	·00787	
				- 768
50	25	2381	·01050	
				869
51	22	2697·5	·00816	
				- 349
52	30	3145	·00954	
				14
53	40	3586	·01115	
				- 287
54	53	4036	·01313	
				44
55	60	4758	·01261	
				532
56	54	5386·5	·01003	
				- 265
57	65	6068·5	·01071	
				- 272
58	81	6749	·01200	
				592
59	84	7511	·01118	
				- 376
60	118	8325·5	·01417	
				- 13

TABLE XV (continued)

Age.	Deaths.	Exposed to Risk.	Rate of Mortality.	Δ^3
61	156	9067	·01721	
				- 457
62	197	9768	·02017	
				884
63	191	10335	·01848	
				- 541
64	227	10818	·02098	
				486
65	263	11813·5	·02226	
				- 863
66	337	12400·5	·02718	
				585
67	347	12800	·02711	
				- 36
68	368	13191	·02790	
				287
69	395	13531	·02919	
				- 164
70	471	13915·5	·03385	
				- 829
71	567	14092	·04024	
				1190
72	567	14152	·04007	
				171
73	631	13948·5	·04524	
				-1921
74	781	13591	·05746	
				1286
75	747	12987	·05752	
				1257
76	720	12355	·05828	
				-1815
77	847	11714	·07231	
				197
78	890	10925	·08146	
				1674
79	833	9945	·08376	
				- 502
80	868	9046	·09595	
				-1845
81	911	8061	·11301	
				1333
82	826	7091	·11649	
				1646
83	737	6156	·11972	
				-1727
84	742	5332	·13916	
				1431
85	696	4418	·15754	
				-6241
86	681	3600	·18917	
				10636

TABLE XV (continued)

Age.	Deaths.	Exposed to Risk.	Rate of Mortality.	Δ^3
87	486	2831·5	·17164	
				-10238
88	482	2281	·21131	
				6538
89	362	1759	·20580	
				- 1980
90	297	1347	·22049	
				933
91	241	1023	·23558	
				4548
92	194	745	·26040	
				- 11615
93	176	517	·34043	
				5796
94	119	331	·35952	
				- 4709
95	74	197	·37563	
				4903
96	41	120	·34167	
				- 7063
97	23	75	·30667	
				43988
98	10	50	·20000	
				- 67024
99	18	39	·46154	
				28511
100	8	19	·42105	
				37736
101	4	11	·36364	
				-133014
102	4	6	·66667	
				213637
103	...	2	......	
				-116667
104	1	2	·5	
105	1	1	1·00000	

With a card of j's for $\epsilon = \cdot 1$ we attached end values 835·83, 802·89, 729·22, 614·82, and 42696, 46753, 50933, 55236. The second differences prove to be constant, thus checking the accuracy. By a difference table we find six graduated values as before and also a sufficient number of external values.

Auxiliary Data.

835·83	and	42696
802·89		46753
729·22		50933
614·82		55236
459·69		59662
263·83		64211
27·24		68883
- 250·08		73678
- 568·13		78596
- 926·91		83637
-1326·42		88801
-1766·66		94088
-2247·63		99498
-2769·33		105031
-3331·76		110687
-3934·92		116466
-4578·81		122368
-5263·43		128393
-5988·78		134541

End Difference Tables.

614·82		
	114·40	
729·22		-40·73
	73·67	
802·89		-40·73
	32·94	
835·83		-40·73
	- 7·79	
828·04		-40·73
	- 48·52	
779·52		-40·73
	- 89·25	
690·27		

55236		
	-4303	
50933		123
	-4180	
46753		123
	-4057	
42696		123
	-3934	
38762		123
	-3811	
34951		123
	-3688	
31263		

To graduate the extended set we used a card of k's to four decimal places. The results were:--

TABLE XV , GRADUATED.

Age.	u	u′	$\Delta^3 u$	$\Delta^3 u'$	u′ − u
40	549	828	1408	30	279
41	1269	780	934	34	− 489
42	697	690	− 2214	19	− 7
43	241	588	1772	20	+ 347
44	835	508	− 297	− 6	− 327
45	265	469	− 517	− 22	+ 204
46	303	491	55	− 28	+ 188
47	652	568	422	− 15	− 84
48	795	678	− 768	− 9	− 117
49	787	793	869	6	+ 6
50	1050	898	− 349	− 7	− 152
51	816	984	14	− 6	+ 168
52	954	1057	− 287	− 3	+ 103
53	1115	1110	44	18	− 5
54	1313	1137	532	32	− 176
55	1261	1135	− 265	21	− 126
56	1003	1122	− 272	11	+ 119
57	1071	1130	592	− 3	+ 59
58	1200	1180	− 376	− 19	− 20
59	1118	1283	− 13	− 28	+ 165
60	1417	1436	− 457	− 13	+ 19
61	1721	1620	884	4	− 101
62	2017	1807	− 541	3	− 210
63	1848	1984	486	3	+ 136
64	2098	2155	− 863	3	+ 57
65	2226	2323	585	22	+ 97

TABLE XV, GRADUATED.(continued)

Age.	u	u'	$\Delta^3 u$	$\Delta^3 u'$	$u' - u$
66	2718	2491	- 36	23	- 227
67	2711	2662	287	17	- 49
68	2790	2858	- 164	- 1	+ 68
69	2919	3102	- 829	- 10	+ 183
70	3385	3411	1190	- 2	+ 26
71	4024	3784	171	- 11	- 240
72	4007	4211	- 1921	- 5	+ 204
73	4524	4690	1286	23	+ 166
74	5746	5210	1257	19	- 536
75	5752	5766	- 1815	- 1	+ 14
76	5828	6381	- 197	- 7	+ 553
77	7231	7074	1674	13	- 157
78	8146	7844	- 502	2	- 302
79	8376	8684	- 1845	15	+ 308
80	9595	9607	1333	32	+ 12
81	11301	10615	1646	1	- 686
82	11649	11723	- 1727	- 87	+ 74
83	11972	12963	1431	- 113	+ 991
84	13916	14336	- 6241	- 58	+ 420
85	15754	15755	10636	99	+ 1
86	18917	17107	-10238	159	- 1810
87	17164	18334	6538	245	+ 1170
88	21131	19535	- 1980	210	- 1596
89	20580	20869	933	55	+ 289
90	22049	22581	4548	- 147	+ 532

TABLE XV, GRADUATED (continued)

Age.	u	u′	$\Delta^3 u$	$\Delta^3 u'$	u′ − u
91	23558	24881	− 11615	− 247	+ 1323
92	26040	27824	5796	− 126	+ 1784
93	34043	31263			− 2780
94	35952	34951			− 1001
95	37563	38762			+ 1199
M_0	427962	428028			
M_2	9·9400	9·9401			

TABLE XV. COMPARISON OF ACTUAL AND EXPECTED DEATHS.

Age.	Exposed to Risk.	Actual Deaths.	Expected.	Deviation.
40	546	3	5	+ 2
41	630·5	8	5	− 3
42	717	5	5	0
43	830	2	5	+ 3
44	958	8	5	− 3
45	1131·5	3	5	+ 2
46	1319·5	4	6	+ 2
47	1533·5	10	9	− 1
48	1760	14	12	− 2
49	2033·5	16	16	0
50	2381	25	21	− 4
51	2697·5	22	27	+ 5
52	3145	30	33	+ 3
53	3586	40	40	0
54	4036	53	46	− 7

TABLE XV. COMPARISON OF ACTUAL AND EXPECTED DEATHS(continued)

Age.	Exposed to Risk.	Actual Deaths.	Expected.	Deviation.
55	4758	60	54	− 6
56	5386·5	54	60	+ 6
57	6068·5	65	69	+ 4
58	6749	81	80	− 1
59	7511	84	96	+ 12
60	8325·5	118	120	+ 2
61	9067	156	147	− 9
62	9768	197	177	− 20
63	10335	191	205	+ 14
64	10818	227	233	+ 6
65	11813·5	263	274	+ 11
66	12400·5	337	309	− 28
67	12800	347	341	− 6
68	13191	368	377	+ 9
69	13531	395	420	+ 25
70	13915·5	471	475	+ 4
71	14092	567	533	− 34
72	14152	567	596	+ 29
73	13948·5	631	654	+ 23
74	13591	781	709	− 72
75	12987	747	749	+ 2
76	12355	720	788	+ 68
77	11714	847	829	− 18
78	10925	890	857	− 33
79	9945	833	864	+ 31

TABLE XV. COMPARISON OF ACTUAL AND EXPECTED DEATHS(continued)

Age.	Exposed to Risk.	Actual Deaths.	Expected.	Deviation.
80	9046	868	869	+ 1
81	8061	911	856	− 55
82	7091	826	831	+ 5
83	6156	737	798	+ 61
84	5332	742	764	+ 22
85	4418	696	696	0
86	3600	681	616	− 65
87	2831·5	486	519	+ 33
88	2281	482	446	− 36
89	1759	362	367	+ 5
90	1347	297	304	+ 7
91	1023	241	255	+ 14
92	745	194	207	+ 13
93	517	176	162	− 14
94	331	119	116	− 3
95	197	74	76	+ 2
M_0		18132	18138	

§19. CRITICISM OF THE RESULTS.

The comparison between the graduated and ungraduated rates of mortality and also between the expected and actual deaths in the two examples just given seems to show that a great gain in regularity has been secured without very great departure from the original data, in spite of the fact that the latter possess several regions of brusque irregularity. Take for example Table XX. The rise and sudden fall visible in the first ten rates is reflected in the graduated rates by a milder sweep, too faithfully perhaps , when the exiguity of the data is considered. Whatever the cause of this curious waviness, it is represented quite definitely in the graduated table and calls for explanation there. A somewhat similar phenomenon is visible in Table XV, except that there the fall occurs quite early, and there is also a second slight dip in the neighbourhood of age 55. It would be interesting to see how these effects were represented by a mode of graduation which took more account of the weights of the observations than we have done. In the remaining part of both graduated tables we find that the deviations are fairly evenly distributed in sign and in magnitude and with a few unavoidable exceptions are not excessively large. The changes of sign of the deviations also fluctuate frequently in an impartial fashion, while the sums of graduated and ungraduated values in all four cases are sensibly identical. That this should be so in the sums of the actual and expected deaths is noteworthy and rather unexpected in its accuracy, for there is nothing in the theory which would have predicted this fact. If it were desired to extend the graduated results beyond age 100 in each case, the values which we have found by extrapolation would almost be suitable; in Table XX they give the rate of mortality as unity at about age 104, and greater than ·6 at and above the age 100; in Table XV they give the rate of mortality as unity at about age 108, and greater than ·6 above age 100.

On the whole, while it is possible that the unreliable rates at high ages have received too much importance, as is inevitable in any method whatever in which the fluctuation of the measure of precision of different observations has not been taken into account, the results of these two graduations may be regarded as very satisfactory.

Restrictions of time have prevented us from including in the present dissertation graduations of the previous tables by the second method which we thought preferable, namely the preliminary adjustment of the deaths proportional to a smoothed exposed to risk, followed by a graduation of the adjusted deaths themselves. It is our purpose to examine the results given by this second method, and also to perform graduations by the methods of this paper on sets of data which have already been graduated by quite different methods, to institute a comparison between the various results, and the respective amounts of time and labour required in obtaining them. It will be essential to do something of the kind, for the results of a few graduations, however good, are not sufficient in themselves to vindicate adequately any new method which may be put forward.

§ 20. REVIEW OF PROGRESS AND CONCLUSION.

We shall conclude with a rapid review of the progress which has been made in the problem before us, and shall consider how far we have gone towards achieving the aims with which we set out originally.

No alteration whatever was made in the basic principle upon which Professor Whittaker founded his theory of Graduation, since we not only regarded this as beyond the province of our research but believed unreservedly in its justness. The à priori hypotheses which Professor Whittaker makes regarding smoothness may be, and have been, criticized; it rests with the critics to express in mathematical terms a better criterion, and then to deduce a method of graduation by a similar use of Bayes' Theorem in Inductive Probability. Our concern, however, was with the mathematical presentation and the practical working out of the principle, and we found it necessary to introduce the following modifications and additions:—

In the purely mathematical formulation and solution of the problem, really of the central difference equation, we adhered as long as possible to the use of Professor Whittaker's Graduation Function G_x, for the simple reason that at first we could not see an alternative. By degrees and only after long and very laborious calculations involving large values of the Graduation Function the conviction was forced upon us that we were in reality attempting to perform the Sisyphean task of making a series with divergent sequence of coefficients a mathematical tool, while all the time there existed an equivalent convergent series. On making this discovery we disburdened ourselves of the Graduation Function, not without great relief. We discovered in fact that the whole mathematical presentation, which turns upon the solution of the central difference equation with constant coefficients

$$\epsilon(u'_x - u_x) = \delta^6 u'_x ,$$

can be simplified and divested of mystery by a straightforward use of symbolic operators: that in the particular case before us it is always possible to find a convergent expansion in powers of operators, of Laurent type: hence that graduated values can be expressed as a series, linear in the ungraduated values on either side, each weighted with one of a sequence of symmetrical coefficients which decreases in practice in such a manner that only a limited number need be employed, the remainder being negligible to the degree of accuracy required. Further, we found that the theory implied the indefinite extension of any given set of crude data by the addition of auxiliary data beyond the termini, the condition being that for these added data graduated and ungraduated values should coincide: and that, if on the one side, let us say the left, of any particular datum all the graduated and ungraduated values were identical, then there existed a second symbolic expansion, of Taylor type, closely related to the the previous and having its successive coefficients decreasing with the same rapidity, by means of which the graduated value of that particular datum was expressed linearly in series in terms of all the ungraduated values on the other side, that is, on the right. Since in practice the number of data requiring to be graduated will be sufficiently large to allow of our using all of the limited number of coefficients retained in this series, we thus discovered a simple means, ϵ being given, of attaching auxiliary data in succession, preparatory to graduating the augmented set by the Laurent operating series. It is to be noted that this method cannot be applied to small sets of data.

Let us digress here to remark that the part of the analysis described in the preceding paragraph can be immediately extended to the case where smoothness is to be measured by the sum of the squares of the r^{th} differences. It may readily be verified, by proceeding in a manner exactly analogous to that of Professor Whittaker's original paper, that the symbolic expansion which then arises for consideration is that of

$$[\,(1-E^{-1})^{2r} + (-)^{r}\epsilon E^{-r}\,]^{-1},$$

and, so long as ϵ is positive, not zero, we can show that

$$[\,(1-z)^{2r} + (-)^{r}\epsilon z^{r}\,]^{-1}$$

is expansible in a series of Laurent type similar to that given earlier in this paper. In addition $(1-z)^{2r} + (-)^{r}\epsilon z^{r}$ may be factorized into two factors of the r^{th} degree, one corresponding to all the roots of modulus less than unity, the other to all the roots of modulus greater than unity, the reciprocals of the first set of roots: if these factors are

$$\frac{1}{a_r}[\,z^{r} - a_1 z^{r-1} + \;\ldots\; +(-)^{r}a_r\,],\ [\,a_r z^{r} - a_{r-1} z^{r-1} + \ldots + (-1)^{r}\,],$$

and the typical coefficient in the expansion in powers of z^{-1} of

$$1 - \frac{(z-1)^r}{z^r - a_1 z^{r-1} + \ldots + (-)^r a_r}$$

is j_s, then the series of powers of the operator E, $\sum_{r=1}^{\infty} j_r E^{-r}$, will furnish a means of extrapolating r auxiliary values at each end, which can be increased to any required number by adding further zero r^{th} differences outwards and building back the table. By this means the process could be transferred at once to a mode of graduation by reduction of irregularities in fourth differences, if that were ever deemed necessary.

We had next to consider the question of what values to assign to ϵ in actual practice in order to secure the most satisfactory balance between smoothness and fidelity. It seemed that the five values ·01, ·05, ·1, ·25 and 1 might well be selected as representative, with ·05 and ·1 as of standard application. A number of reasons combined to dictate such a selection. First the selection of fixed discrete values is justified by the fact, observed from actual results of graduation, that small changes in ϵ produce quite small changes in graduated values, so that graduations performed, e.g. with ϵ = ·08, ·09, ·1, ·11, ·12, are so little different that ϵ = ·1 may well be taken to represent this region. Then comparison of the "Laurent" coefficients corresponding to ϵ = ·01 and ϵ = ·05 with the weights corresponding to the well known Spencer formula of graduation showed that the magnitude and fluctuation of the latter would be nearly the same, at least in the initial large coefficients, as if ϵ were about ·02. Since the present theory was expected to harmonize fidelity with smoothness to a greater degree than the Spencer formula, it seemed justifiable to take a standard value of ϵ somewhat larger than ·02, namely ·05 or even ·1, according to the regularity of the data to be graduated. The values ·25 and 1 were included for completeness, although it is extremely improbable that either of them, and especially 1, will ever be of great practical service.

For these five values of ϵ we calculated with the aid of machines the successive coefficients of the extrapolating and graduating series associated with each value, to six places of decimals in every case, a greater degree of accuracy than would ever be required, so that for practical use they could be cut down from these standard tables to the number of decimal places considered sufficient. There seems much to be said for reducing all statistics of mortality rates and the like to no more than three significant digits and for graduating these by coefficients to three or four digits. Attempts to preserve a higher number of digits merely render the work more difficult and laborious to perform, while aiming at an ultimate degree of accuracy which is quite fanciful when one considers the nature of the crude material presented for graduation.

We endeavoured to reduce the use of the two series to a mechanical routine in the following order:—

(i) Write down the data to be graduated and form a table of differences as far as the third.

(ii) Examine the middle part of the sequence of third differences and observe where the largest and most irregular ones occur. Apply to one or two of these the graduating series for $\epsilon = \cdot 1$ or $\epsilon = \cdot 05$, in order to observe by the result what reduction in third differences has been effected. From such trials (for which only a few minutes are required) fix upon the value of ϵ to be adopted.

(iii) Extrapolate at each end four external data, using a card of coefficients after the manner of §18, and summing products on an arithmometer.

(iv) Form a difference table for the four added data at each end. As a check the single third difference in each case should be zero. If it is, insert three more zero third differences inwards and as many outwards as the convergence of the graduating series, i.e. the number of coefficients used, demands; then build back the table by summation. The inner three values at each end give us six graduated values, which may be entered at once in the result column; the rest are to be added to the ungraduated set.

(v) With a card of graduating coefficients find the rest of the graduated values as in §18.

We carried out this process on two mortality tables of 55 and 56 entries respectively and found that the method worked satisfactorily and smoothly and that the results were good.

We failed to achieve success in the following respects;-

It is highly desirable to leave the "exposed to risk" in as an influencing factor on graduated rates of mortality, etc., but we could not reduce to a practicable form the solution of the more general equation with variable coefficients

$$\epsilon_x(u'_x - u_x) = \delta^6 u'_x ,$$

where ϵ_x is now dependent on x. Hence we were forced to continue to assume ϵ as constant in our work, and were left with the alternatives, either to discard a certain number of unreliable data at the ends and to graduate the remainder, reserving the ends for later consideration, or to graduate or smooth the exposed to risk, alter the actual deaths proportionally, then graduate the deaths themselves and thus find the graduated rate of mortality as the quotient of graduated deaths divided by the altered exposed to risk. The second alternative is evidently preferable, but is the more tedious in requiring more operations.

In conclusion we may record a certain measure of success in having so modified the mathematical presentation as to yield a method of graduation in every way more practicable than the original and applicable to extended sets of data without any disproportionate increase of labour and time, but a lack of success in our efforts to extend the method to cases where, as always happens, the degree of precision of measurement varies between one datum and the next.

Edinburgh.

May, 1925.